

General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.
- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.
- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.
- This document is paginated as submitted by the original source.
- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

(NASA-CR-150317) MODEL VERIFICATION OF
LARGE STRUCTURAL SYSTEMS Interim Report
(Wiggins (J. H.) Co., Redondo Beach, Calif.)
121 p HC A06/MF A01 CSCL 20K

N77-27428

G3/39 Unclass
36697



j.h.wiggins company

MODEL VERIFICATION OF LARGE STRUCTURAL SYSTEMS
TECHNICAL REPORT 77-1267 • PREPARED FOR GEORGE C. MARSHALL SPACE FLIGHT CENTER, NATIONAL AERONAUTICS AND SPACE ADMIN.



**MODEL VERIFICATION
OF
LARGE STRUCTURAL SYSTEMS**

INTERIM REPORT

Technical Report 77-1267

Prepared for:

**National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama**

Prepared by:

**L. T. Lee
T. K. Hasselman**

**J. H. Wiggins Company
1650 South Pacific Coast Highway
Redondo Beach, California 90277**

June 1977

TABLE OF CONTENTS

	<u>PAGE</u>
1. INTRODUCTION	1
2. DERIVATION OF ESTIMATOR EQUATIONS	4
2.1 The Minimum Variance Estimator	4
2.2 Mathematical Models	6
2.3 Equations of Motion	
2.4 Frequency Response Measurements	9
2.5 Sensitivity Matrix Relating Observation and State Vectors for Phase I	13
2.6 Modification of the Analytical Model (Phase I)	24
2.7 Sensitivity Matrix Relating Observation and State Vectors for Phase II	26
2.8 Modification of the Dynamic Model (Phase II)	32
3. EXAMPLE PROBLEMS	34
3.1 One Degree-of-Freedom Problem	36
3.2 Two Degree-of Freedom Problem	39
3.3 Three Degree-of-Freedom Problem	48
3.4 Two Degree-of-Freedom Problem: Close Modes	53
4. METHOD FOR IMPROVING PRIOR ANALYTIC MODEL	57
4.1 First Order Correction	57
4.2 Criteria for Banding $[m]$ and $[k]$	63
4.3 Sample Calculations	67
5. DEFINITION OF SUB-MATRICES	71
5.1 Substructing Basis	71
5.2 Orthogonal Modes Basis	72

	<u>PAGE</u>
6. CONCLUSIONS AND RECOMMENDATIONS	76
6.1 Conclusions	76
6.2 Recommendations	77
References	80
Nomenclature	81
APPENDIX A. Description of Computer Program	A-1
A-1 Program ESTIMA (Phase I)	A-2
A-2 Program ESTIMB (Phase II)	A-19

FOREWORD

The material documented in this interim report describes progress to date accomplished under NASA Contract No. NAS8-31950, during the period 13 April 1976 through 14 June 1977. During this period, a methodology has been formulated, and a general computer code implemented and checked out, for processing sinusoidal vibration test data to simultaneously make adjustments to a prior mathematical model of a large structural system, and resolve measured response data to obtain a set of orthogonal modes representative of the test article.

The general procedure is referred to as "model verification". The term "model verification" is used herein to denote a procedure with two distinct and equally important objectives: (1) to establish a proper model configuration by examining different variations of configuration with respect to their ability to match available test data, and (2) to estimate specific parameter values for a selected model configuration. The first objective is met by providing a general modeling capability within the logical structure of the computer code. The practical utility provided by this modeling capability is intended to facilitate a "man-in-the-loop" type of function, where the user may apply his judgement and modeling skill to achieve a proper model configuration. The second objective is met by providing fully automated parameter estimation programs to optimize the fit of any selected model configuration to the given test data.

Consistent with the "man-in-the-loop" approach, the computer code is implemented in two separate programs, where the output from one becomes input to the other. Thus, the user may inspect output from the first (frequencies, damping, mode

shapes and sinusoidal response) prior to completing computations to evaluate revised mass and stiffness matrices, and the changes which have been made to the prior mathematical (matrix) model.

During the continuing part of this contract, modeling and computational procedures will be refined to improve the overall capability of the procedure to deal with real-world problems, and to provide user-oriented guidelines for application of the methodology.

1. INTRODUCTION

The J. H. Wiggins Company has developed, under contract to NASA-MSFC, a methodology and computer program for structural system identification. This work addresses a general need in the area of analysis-test correlation with particular focus on the Space Shuttle Program. The objective of this effort is to enable computerized processing of sinusoidal vibration test data to refine pre-established analytical models. The two primary objectives of the effort are: (1) to develop and check out the computer code itself; and (2) to demonstrate its application to a real problem, i.e., using actual test data and a mathematical model for a real structural system.

The first objective has been achieved. A computer code has been developed and checked out using simple analytical models and simulated test data. Several analytical models have been investigated in an effort to gain basic experience with the parameter estimation procedure, and to examine numerical sensitivity to different types of parameters. This experience has verified the basic operation of the code. In addition, it has pointed to some areas where improvement to the existing code is desirable from the standpoint of practical application to real structural systems.

The proposed demonstration problem has not been undertaken. It was originally proposed that the full-scale orbiter and HGVT data be used. Although the present computer code can be applied to this problem, it became apparent that the overall complexity of the problem, considering the dynamic model as well as the HGVT data, is such that the likelihood of achieving the second objective is not as good as originally

hoped. Consequently a different tack is proposed: Select a simpler demonstration problem first and gradually increase problem complexity until the full-scale orbiter data can be evaluated.

Three basic steps are involved in the present computerized methodology with an additional step being recommended here.

- 1) Establish a suitable prior model, including both a modal representation (the analytic model) and mass and stiffness representation in terms of the physical coordinates (the dynamic model).

(NEW) Perform a "zero" order identification using measured natural frequency and mode shape data.

- 2) Refine the parameters of the analytic model using measured response data.
- 3) Refine the parameters of the dynamic model using the refined analytic model as if it were test data.

It has been determined that the "closeness" of the prior model to the actual system is an important factor in the convergence behavior of the estimator. This is especially true in the case of the Orbiter where response data are only available at resonant frequencies. Such data will constitute a rather sparse sampling of frequency response characteristics, tending to make convergence more difficult to achieve.

It has been determined that some additional effort will be required to come up with a good "starting point" for the prior model. In particular, the test frequencies and cross-orthogonality coefficients between analytical and test modes

can be used to obtain an improved prior modal model, based on a first-order "correction" of the original prior modal model. This is the "new" operation to be implemented. It will be discussed in more detail in Section 4.

The procedure planned for processing response data in the first stage of estimation will entail sequential use of data. Briefly, we plan to work within selected frequency bands as we march along the frequency spectrum. Within each band, we will process sets of response data selectively.

In the second stage of estimation where we proceed from the refined modal model to update the mass and stiffness matrices of the "dynamic model", the critical and difficult step will be to define the submatrices to be associated with the scaling parameters. The operation will be discussed in Section 5.

2. DERIVATION OF ESTIMATOR EQUATIONS

This section presents the derivation of all of the equations which make up this identification scheme except those which make up the minimum variance estimator. The derivation of those equations is described in References 1 and 2. The computer code which implements the methodology is described in Appendix A.

Two separate computer programs have been developed: One to perform the Phase I estimation of the analytic model and the other to perform the Phase II estimation of the dynamic model. Presently the user must start each program with card data. This will be changed later to make the data interface automatic using disk or tape storage. The programs will not be automatically coupled, however, because the user will always need to evaluate the results of Phase I before proceeding to Phase II. It may, in fact, take several Phase I runs using different data and/or estimating different parameters before an engineering-sound model is obtained. The methodology does not substitute for a sound understanding of the structure. It is only a tool. The better the user understands the structure, the more useful the tool will be -- and visa versa -- the more he uses the tool, the better he will understand the behavior of the structure.

2.1 The Minimum Variance Estimator

The model adjustments provided by this methodology are made with the minimum variance estimator described in References 1 and 2. This is a linear estimator in that it operates on the basis of a linearized relationship between a state vector $\{x\}$

and an observation vector $\{Y\}$. If r_p and y_p denote prior estimates of these vectors based upon analytical models, then the true values r and y are assumed to be related by the linear matrix equation

$$\{Y - y_p\} = [T] \{r - r_p\} \quad (2-1)$$

If $\{Y\}$ represents measured data with random error $\{\epsilon\}$ and the simplified notation, $\{Y - y_p\} = \{y\}$ and $\{r - r_p\} = \{R\}$ is adopted, then equation 2-1 becomes

$$\{y\} = [T] \{R\} + \{\epsilon\} \quad (2-2)$$

as shown in equation 24 of Reference 2. The derivation of the statistical estimator equations beginning on page 187 of Reference 2 applies here also, except for the following differences

- The analytical model is a modal representation of the structure, not a finite element representation.
- The state vector may contain elements of the damping matrix or it may contain elements of the mass and stiffness matrices.
- The measured data is the amplitude of the displacement response measured at discrete frequencies and locations and due to a specified set of forces.

In essence, this formulation first tries to refine the modal parameters of the analytical model while at the same time acting as a filter on the measured data to overcome experimental difficulties. By using frequency-response data as the

basis for comparison it is not necessary to excite "pure" modes. The use of frequency-response data also allows the estimation of elements of the damping matrix.

2.2 Mathematical Models

Before proceeding with the formulation of the minimum variance estimator for this parameter identification approach, it is necessary to summarize the hierarchy of mathematical models used to represent the structural system and to define our notation. Four models are germane to the present discussion:

- Finite Element Model - This model represents the most detailed model of the system. Created part by part, it may contain up to 100,000 degrees of freedom depending on the size, complexity, and importance of the structure being analyzed.
- Dynamic Model - This model is obtained by taking the finite element model, or the components of that model, through several stages of coordinate reduction. This model may contain up to 750 degrees of freedom. At least some of these degrees of freedom must relate directly to physical structural displacements so that a comparison to measured test data can be made. In the development that follows this model is referred to as the "u" system.
- Analytical Model - This model is obtained by computing a limited number of modes of the dynamic model and using them as the basis for an additional coordinate reduction. Very often for spacecraft structures all modes of the dynamic model (i.e., the

u system) below 50 Hz are selected for use in the analytical model. This may result in as many as 200 and as few as 20 modes being used. The size (i.e., number of degrees of freedom) in the analytical model is the number of modes being used. These are the modes which should and can be verified with vibration test data. In the development that follows, this model is referred to as the "x" system.

- Modal Model - Prior to the Phase I estimation, the modal model does not exist because the coordinate reduction described above diagonalizes the dynamic model mass and stiffness matrices. There is then no need for another transformation. However, once an estimation has been performed the analytic model mass and stiffness matrices are no longer diagonal. Now a second coordinate transformation can be defined using the modes of the perturbed analytic model. The resulting coordinates are herein referred to as the modal model or the "q" system.

2.3 Equations of Motion

Let the displacements of the dynamic model be represented by the vector $\{u(t)\}$. Then the equations of motion for this system can be represented by

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{f(t)\} \quad (2-3)$$

where

$[M]$ = mass matrix of dynamic model

$[K]$ = stiffness matrix of dynamic model

$[C]$ = damping matrix of dynamic model
 $\{f(t)\}$ = vector of forcing functions

For our purposes $\{f(t)\}$ may be further broken down into a vector characterizing the force distribution $\{P\}$ and a scalar time function, $g(t)$

$$\begin{aligned}\{f(t)\} &= \{P\} g(t) \\ &= \{P\} g \sin \Omega t\end{aligned}\tag{2-4}$$

The undamped eigenvalue problem for this system may be stated as

$$([K] - \lambda_j [M]) \{\phi_j\} = 0\tag{2-5}$$

Its solution will yield a set of eigenvalues $[\lambda]$ and eigenvectors $[\phi]$. Using this eigenvector matrix, or a truncated portion of it, as a coordinate transformation, results in the analytical model. For our purposes, we will assume that the mode shapes $[\phi]$ have been normalized for unit generalized mass. We will now introduce the left-hand superscript o to represent the matrices associated with the prior dynamic model before any changes have been implemented. Letting

$$\{u\} = [^o\phi] \{x\}\tag{2-6}$$

and premultiplying by $[^o\phi]^T$ transpose, equation 2-3 becomes

$$\begin{aligned}&[^o\phi]^T [^oM] [^o\phi] \{\ddot{x}\} + [^o\phi]^T [^oC] [^o\phi] \{\dot{x}\} \\ &+ [^o\phi]^T [^oK] [^o\phi] \{x\} = [^o\phi]^T \{P\} g(t)\end{aligned}\tag{2-7}$$

PRECEDING PAGE CLAIM NOT FILMED

PRECEDING PAGE CLAIM NOT FILMED

but

$$\begin{aligned} [\phi]^T [M] [\phi] &= [I] \text{ diagonal} \\ &= \text{generalized mass matrix} \\ [\phi]^T [C] [\phi] &= [c] \\ &= \text{generalized damping matrix} \\ [\phi]^T [K] [\phi] &= [\lambda] \text{ diagonal} \\ &= \text{generalized stiffness matrix} \\ [\phi]^T \{P\} &= \{\tilde{P}\} \end{aligned}$$

as a result of the modal normalization and the properties of the modal matrix. This results in the equations of the initial analytical model:

$$[I] \{\ddot{x}\} + [c] \{\dot{x}\} + [\lambda] \{x\} = \{\tilde{P}\} g(t) \quad (2-8)$$

This set of equations will now be defined as the prior analytic model where $[I]$ and $[\lambda]$ are diagonal matrices and all of the matrices plus $[\phi]$ are a direct result of the unmodified dynamic model.

The objective of the Phase I parameter estimation is to develop revised mass and stiffness matrices for the analytic model and a revised generalized damping matrix for the modal model which will, when incorporated into the response analysis, provide an improved correlation between the calculated responses and the measured responses. Before preceding to a further discussion of the parameter estimation, let us define the terms used in the response analysis.

Once the analytic model diagonal mass and stiffness matrices have been perturbed they will no longer be diagonal. Now a second transformation can be defined to diagonalize these matrices. Let

$[\psi]$ = modes of perturbed analytic model
 $[m]$ = perturbed mass matrix - analytic model
 $[k]$ = perturbed stiffness matrix - analytic model

and the transformation be

$$\{x\} = [\psi] \{q\}$$

The equations of motion now become

$$[I] \{\ddot{q}\} + [\xi] \{\dot{q}\} + [\lambda] \{q\} = [\psi]^T \{\tilde{P}\} g(t)$$

where

$$[\psi]^T [m] [\psi] = [I] = [\psi]^T [\phi] [M] [\phi] [\psi]$$

$$[\psi]^T [c] [\psi] = [\xi] = [\psi]^T [\phi] [C] [\phi] [\psi]$$

$$[\psi]^T [k] [\psi] = [\lambda] = [\psi]^T [\phi] [K] [\phi] [\psi]$$

Only $[I]$ and $[\lambda]$ are diagonal.

2.4 Frequency Response Measurements

The displacement response of the dynamic system, denoted by $\{u\}$, can be represented as a transfer function between the response $\{u\}$ and the forcing function $\{P\}g(t)$.

$$\{u(t)\} = [H(\Omega)]\{P\}g(t)$$

$[H(\Omega)]$ also performs the same task if the problem is transformed from the time domain to the frequency domain.

$$\{U(\Omega)\} = [H(\Omega)] \{P\} G(\Omega)$$

Furthermore, the transfer function matrix, $[H(\Omega)]$, for the u system can be expressed as a function of the eigenvectors (i.e., mode shapes) $[\phi]$, the eigenvalues $[\lambda]$, and the damping matrix $[c]$

$$[H(\Omega)] = [\phi] [\tilde{H}(\Omega)] [\phi]^T \quad \text{matrix notation}$$

or

(2-9)

$$H_{ik} = \sum_{q,m} \phi_{iq} \tilde{H}_{qm} \phi_{km} \quad \text{indicial notation}$$

where

$$[H(\Omega)] = \left[-[I]\Omega^2 + [c]\Omega\hat{i} + [\lambda] \right]^{-1} = [\tilde{Z}(\Omega)]^{-1} \quad (2-10)$$

$$\begin{aligned} \text{Derivation: } \{u\} &= [\phi] \{x\} \\ &= [\phi] [\tilde{H}(\Omega)] \{\tilde{P}\} g(t) \\ &= [\phi] [\tilde{H}(\Omega)] [\phi]^T \{P\} g(t) \\ &= [H(\Omega)] \{P\} g(t) \end{aligned}$$

We have switched from the superscript "o" to no superscript to indicate that the model parameters to be used are those for the dynamic model as it stands at the time. Only at the beginning of the iterative procedure will $[\phi]$ and $[\lambda]$ be identical to $[^o\phi]$ and $[^o\lambda]$ respectively.

Excitation frequencies Ω_j need not correspond to resonant frequencies but refer to any selected excitation frequency. The modulus of the complex frequency response will be used as a comparative basis for computed and measured response in order to confine the estimator to operate in real arithmetic.

To illustrate the physical significance of the new observation vector, Figures 1 and 2 are provided. Figure 1 shows a schematic representation of a free-free beam excited laterally at one end by a sinusoidal force $g(t) = g \sin \Omega t$. The beam is modeled with five transverse degrees of freedom. Three non-rigid-body modes and frequencies can be obtained from this model. Depending on the stiffness and mass distribution, some of these may be closely spaced. The frequency response characteristics over a range including the first two frequencies may resemble those depicted in Figure 2. Four discrete excitation frequencies are identified in Figure 2. In this case the effective observation vector $\{y\}$ contains 20 elements including four vectors $\{u\}$ each containing five elements. Although a relatively complete set of response measurements is indicated here, it is emphasized that any number of measurements may be considered. For example, it may be sufficient to include only four out of the 20 corresponding to only two response measurements at u_1 and u_5 and two excitation frequencies Ω_1 and Ω_4 .

Intuitively, one set of response locations and excitation frequencies may be more suitable for calculations than another. Experienced judgment will be helpful in making a good selection. However, different sets of measurement data may be processed sequentially by the estimator so that one need not be restricted to choosing a single optimum set of data.

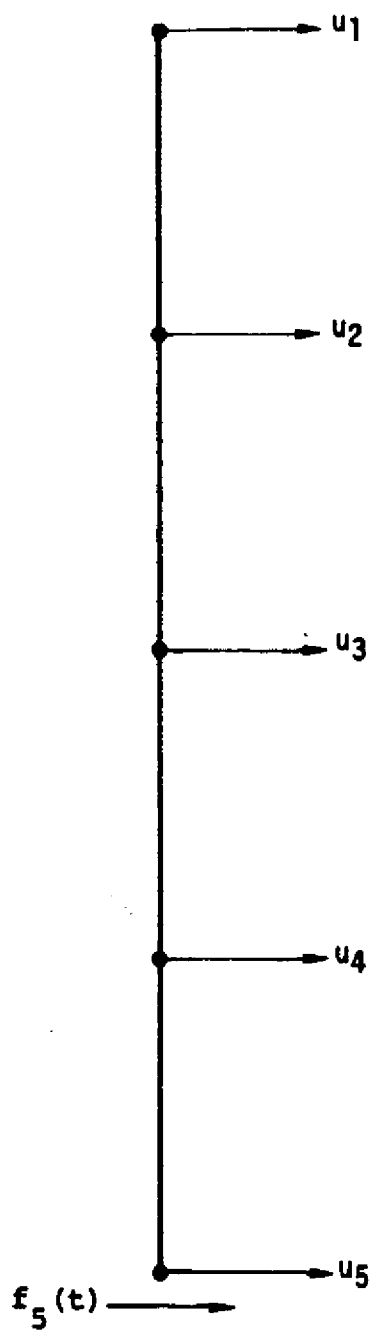


Figure 1. Finite Element Model of a Free-Free Beam

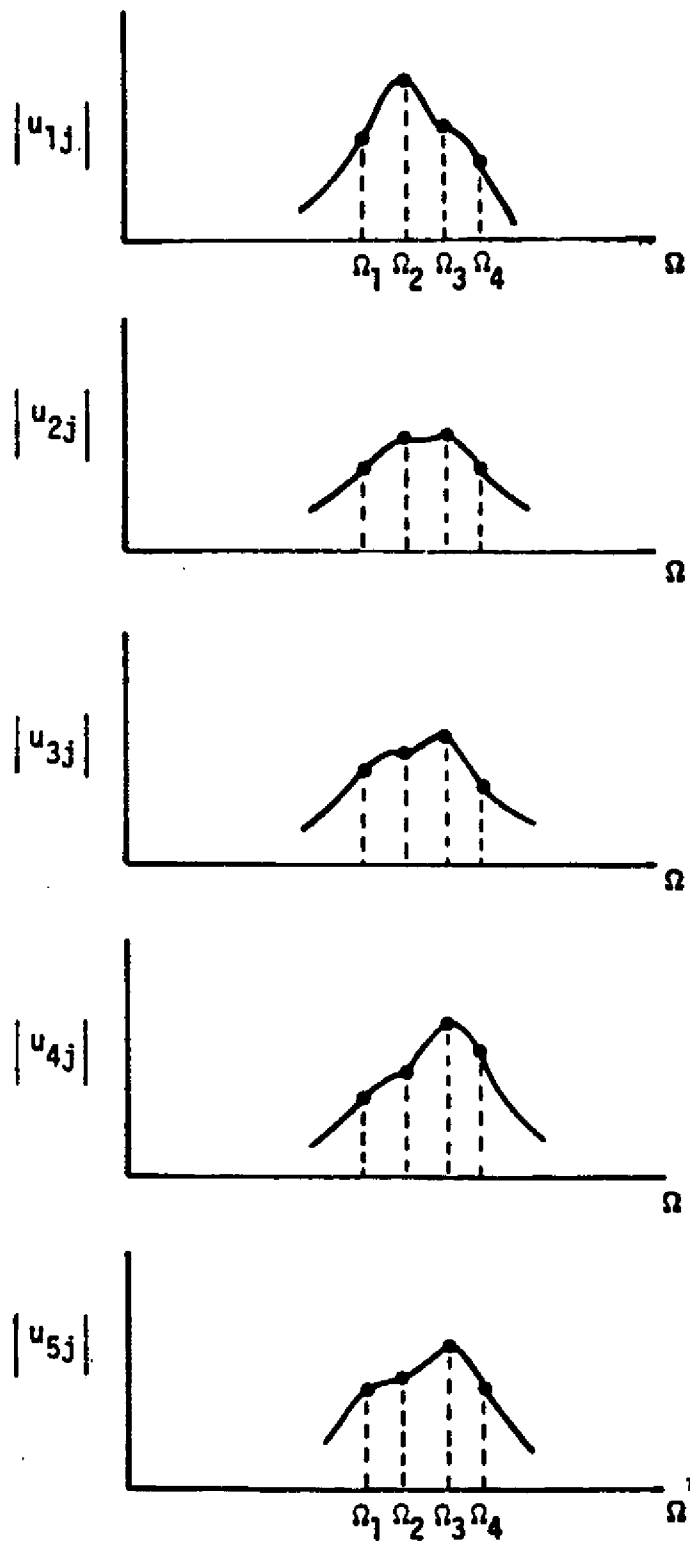


Figure 2. Frequency Response of Model Shown in Figure 1

2.5 Sensitivity Matrix Relating Observation and State Vectors for Phase I

For the system identification methodology developed here, the observation vector is any set of discrete response measurements which has been chosen as data. Furthermore only the modulus (i.e., magnitude) of the measured responses is being used. The state vector, on the other hand, is composed of the components of the generalized mass matrix and the generalized stiffness matrix. Initially the generalized mass matrix is unity and the generalized stiffness is a diagonal matrix of the eigenvalues of the prior dynamic model. The sensitivity matrix [T] relates the state vector to the observation vector. The derivation of this matrix is shown below and summarized in Table 1. First the primary elements in the nomenclature need to be defined:

$$Y_i \equiv |u_i| = \text{magnitude of } i^{\text{th}} \text{ response} \quad (2-12)$$

$$= i^{\text{th}} \text{ component of observation vector}$$

$$H_{ik}(\Omega) \equiv A_{ik} + B_{ik}f = i, k \text{ element of } [H(\Omega)] \quad (2-13)$$

$$P_k = R_{P_k} + I_{P_k} \quad (2-14)$$

$$u_i = R_{u_i} + I_{u_i} f \quad (2-15)$$

$$u_i = \sum_k H_{ik} P_k$$

$$= \sum_k (A_{ik} R_{P_k} - B_{ik} I_{P_k}) + f \sum_k (A_{ik} I_{P_k} + B_{ik} R_{P_k}) \quad (2-16)$$

Y_i , the magnitude of u_i is given by

$$Y_i = (u_i \ u_i^*)^{1/2} \quad (2-17)$$

Table 1
SUMMARY OF OPERATIONS REQUIRED
TO DETERMINE SENSITIVITY MATRIX

	Equation No.
A) $T_{lh} = \frac{\partial Y_l}{\partial r_h}$	(2-19)
B) $\frac{\partial Y_l}{\partial r_h} = [u_i \ u_{i*}]^{-1/2} \left[R_{u_i} \frac{\partial R_{u_i}}{\partial r_h} + I_{u_i} \frac{\partial I_{u_i}}{\partial r_h} \right]$	(2-21)
C) $\frac{\partial R_{u_i}}{\partial r_h}$ and $\frac{\partial I_{u_i}}{\partial r_h}$ require $\frac{\partial H_{ik}}{\partial r_h}$	(2-22)
D) $\frac{\partial H_{ik}}{\partial r_h}$ requires $\frac{\partial \psi_{nq}}{\partial r_h}$ and $\frac{\partial \tilde{H}_{qm}}{\partial r_h}$	(2-25, 26)
E-1) $\frac{\partial \tilde{H}_{qm}}{\partial r_h} = - \sum_{l,l} \tilde{H}_{qi} \left(\delta_{il} \frac{\partial \lambda_{il}}{\partial r_h} + \Omega \frac{\partial \xi_{il}}{\partial r_h} \right) \tilde{H}_{lm}$	(2-27)
E-2) $\frac{\partial \psi_{ij}}{\partial r_h} = \sum_k \psi_{ik} \frac{\partial \eta_{kj}}{\partial r_h}$ where η defined Reference 6	(2-37)
F-1) $r_h = k_{rs} : \frac{\partial \lambda_i}{\partial k_{rs}} = \psi_{ri} \psi_{si}$	(2-28)
$\frac{\partial \xi_{il}}{\partial k_{rs}} = \begin{cases} 0 \text{ (neglect)} & i \neq l \text{ (2-31)} \\ \zeta_i \lambda_i^{-1/2} \frac{\partial \lambda_i}{\partial k_{rs}} m_{ii} & i = l \text{ (2-32)} \end{cases}$	

Table 1 (Continued)

$$\frac{\partial \eta_{kj}}{\partial k_{rs}} = \sum_{\ell, m} \psi_{\ell k} \left(\frac{1 - \delta_{kj}}{\lambda_j - \lambda_k} \right) \delta_{\ell r} \delta_{ms} \psi_{mj} \quad \text{Equation No. (2-38)}$$

$$F-2) \quad r_h = m_{rs} : \quad \frac{\partial \lambda_i}{\partial m_{rs}} = -\lambda_i \psi_{ri} \psi_{si} \quad (2-29)$$

$$\frac{\partial \xi_{il}}{\partial m_{rs}} = \begin{cases} 0 \text{ (neglect)} & i \neq l \\ \zeta_i \lambda_i^{-1/2} \frac{\partial \lambda_i}{\partial m_{rs}} m_{ii} & i = l \end{cases} \quad (2-31)$$

$$\frac{\partial \eta_k}{\partial m_{rs}} = \sum_{\ell, m} \psi_{\ell k} \left[\left(\frac{1 - \delta_{kj}}{\lambda_j - \lambda_p} \right) \left(-\lambda_i \delta_{\ell r} \delta_{ms} \right) - \frac{\delta_{kj} \delta_{\ell r} \delta_{ms}}{2} \right] \psi_{mj}$$

$$F-3) \quad r_h = \xi_{rs} : \quad \frac{\partial \lambda_i}{\partial \xi_{rs}} = 0 \quad (2-30)$$

$$\frac{\partial \xi_{il}}{\partial \xi_{rs}} = \delta_{ir} \delta_{ls} \quad (2-33)$$

$$\frac{\partial \eta_k}{\partial \xi_{rs}} = 0 \quad (2-42)$$

Notes: 1) The state vector will have m_{rs} and k_{rs} or ξ_{rs} terms - never both at the same time since the methodology has been set up to estimate ξ_{rs} separately.

2) $\lambda_{ii} \equiv \lambda_i$

where the * refers to the complex conjugate, the superscripts R and I refer to the real and imaginary components respectively, and the superscript ~ refers to the u system (i.e., dynamic model).

$$\tilde{H}_{qm} = a_{qm} + b_{qm} \hat{i} \quad (2-18)$$

The elements of [T] are given by

$$T_{lh} = \frac{\partial Y_l}{\partial r_h} \quad (2-19)$$

where the index l denotes the lth element of the observation vector Y and the index h denotes the h element of the state vector r. Taking the partial derivative of Y using equation 2-17:

$$\frac{\partial Y_l}{\partial r_h} = \frac{1}{2} [u_i \ u_i^*]^{-1/2} \left[\frac{\partial u_i}{\partial r_h} u_i^* + u_i \frac{\partial u_i^*}{\partial r_h} \right] \quad (2-20)$$

but

$$\frac{\partial u_i}{\partial r_h} u_i^* = R_{u_i} \frac{\partial R_{u_i}}{\partial r_h} + I_{u_i} \frac{\partial I_{u_i}}{\partial r_h}$$

$$u_i \frac{\partial u_i^*}{\partial r_h} = R_{u_i} \frac{\partial R_{u_i}}{\partial r_h} + I_{u_i} \frac{\partial I_{u_i}}{\partial r_h}$$

Therefore

$$\frac{\partial Y_l}{\partial r_h} = [u_i, u_i^*]^{-1/2} \left[R_{u_i} \frac{\partial R_{u_i}}{\partial r_h} + I_{u_i} \frac{\partial I_{u_i}}{\partial r_h} \right] \quad (2-21)$$

where R_{u_i} and I_{u_i} are defined in terms of the real and imaginary components of the transfer function $H_{ik}(\Omega)$ and the force vector u_{p_k} by equation 2-16. Thus

$$\frac{\partial R_{u_i}}{\partial r_h} = R_{p_k} \frac{\partial A_{ik}}{\partial r_h} - I_{p_k} \frac{\partial B_{ik}}{\partial r_h}$$

$$\frac{\partial I_{u_i}}{\partial r_h} = I_{p_k} \frac{\partial A_{ik}}{\partial r_h} + R_{p_k} \frac{\partial B_{ik}}{\partial r_h}$$

Using the transformation discussed later (Section 2.7)

$$[\phi] = [{}^o\phi] [\psi] \quad \text{matrix notation}$$

or

$$\phi_{iq} = \sum_n {}^o\phi_{in} \psi_{nq} \quad \text{indicial notation} \quad (2-23)$$

The elements of the transfer function H_{ik} can be expressed as

$$[H] = [{}^o\phi] [\psi] [\tilde{H}] [\psi]^T [{}^o\phi]^T \quad \text{matrix notation} \quad (2-24)$$

or

$$H_{ik} = \sum_{n,q,m,o} {}^o\phi_{in} \psi_{nq} \tilde{H}_{qm} {}^o\phi_{ko} \psi_{om} \quad \text{indicial notation}$$

Thus

$$\begin{aligned}
 \frac{\partial A_{ik}}{\partial r_h} &= \text{derivative of real part of equation 2-13} \\
 &= \sum_{n,q,m,o} {}^\circ\phi_{in} \frac{\partial \psi_{nq}}{\partial r_h} {}^{R\tilde{H}}_{qm} {}^\circ\phi_{ko} \psi_{om} \\
 &\quad + \sum_{n,q,m,o} {}^\circ\phi_{in} \psi_{nq} \frac{\partial {}^{R\tilde{H}}_{qm}}{\partial r_h} {}^\circ\phi_{ko} \psi_{om} \\
 &\quad + \sum_{n,q,m,o} {}^\circ\phi_{in} \psi_{nq} {}^{R\tilde{H}}_{qm} {}^\circ\phi_{ko} \frac{\partial \psi_{om}}{\partial r_h} \quad (2-25)
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{\partial B_{ik}}{\partial r_h} &= \text{derivative of imaginary part of equation 2-13} \\
 &= \sum_{n,q,m,o} {}^\circ\phi_{in} \frac{\partial \psi_{nq}}{\partial r_h} {}^{I\tilde{H}}_{qm} {}^\circ\phi_{ko} \psi_{om} \\
 &\quad + \sum_{n,q,m,o} {}^\circ\phi_{in} \psi_{nq} \frac{\partial {}^{I\tilde{H}}_{qm}}{\partial r_h} {}^\circ\phi_{ko} \psi_{om} \\
 &\quad + \sum_{n,q,m,o} {}^\circ\phi_{in} \psi_{nq} {}^{I\tilde{H}}_{qm} {}^\circ\phi_{ko} \frac{\partial \psi_{om}}{\partial r_h} \quad (2-26)
 \end{aligned}$$

Both $[\phi]$ and $[\psi]$ are real matrices. Only $[\tilde{H}]$ is complex. To complete the evaluation of T_{lh} requires the determination of $\frac{\partial \tilde{H}_{qm}}{\partial r_h}$ and $\frac{\partial \psi_{nq}}{\partial r_h}$. To obtain the first of these, first dif-

ferentiate equation 2-10 and then consider the three different variables that r_h may be.

$$\begin{aligned} \frac{\partial [\tilde{H}(\Omega)]}{\partial r_h} &= (-1) [\tilde{Z}(\Omega)]^{-2} \left[\frac{\partial \tilde{Z}(\Omega)}{\partial r_h} \right] \\ &= - [\tilde{H}(\Omega)] \left[\frac{\partial \tilde{Z}(\Omega)}{\partial r_h} \right] [\tilde{H}(\Omega)] \end{aligned} \quad (2-27)$$

or

$$\frac{\partial \tilde{H}_{qm}}{\partial r_h} = - \sum_{i,l} \tilde{H}_{qi} \left(\delta_i \frac{\partial \lambda_i}{\partial r_h} + \Omega \frac{\partial \xi_{il}}{\partial r_h} \hat{i} \right) \tilde{H}_{lm}$$

The parameters being estimated are the elements of the analytic model stiffness matrix, denoted by k_{rs} ; the elements of the analytic model mass matrix, denoted by m_{rs} ; and the elements of the generalized damping matrix, denoted by ξ_{rs} . Only k_{rs} and m_{rs} directly influence the new model. Although ξ_{rs} has an indirect effect, its estimation is a separate operation. ξ_{rs} is part of the modal model while m_{rs} and k_{rs} are part of the analytic model. First consider $\frac{\partial \lambda_i}{\partial r_h}$

$$\frac{\partial \lambda_i}{\partial r_h} = \sum_{j,l} \left(\psi_{ji} \frac{\partial k_{jl}}{\partial r_h} \psi_{li} - \lambda_i \psi_{ji} \frac{\partial m_{jl}}{\partial r_h} \psi_{li} \right) \quad [3]$$

$$\text{CASE 1) } r_h = k_{rs} : \frac{\partial k_{jl}}{\partial k_{rs}} = \delta_{jr} \delta_{ls} \quad \text{and} \quad \frac{\partial m_{jl}}{\partial k_{rs}} = 0$$

then

$$\frac{\partial \lambda_i}{\partial k_{rs}} = \sum_{j,l} \psi_{ji} \delta_{jr} \delta_{ls} \psi_{li} = \psi_{ri} \psi_{si} \quad (2-28)$$

$$\text{CASE 2) } r_h = m_{rs} : \frac{\partial k_{jl}}{\partial m_{rs}} = 0 \quad \text{and} \quad \frac{\partial m_{jl}}{\partial m_{rs}} = \delta_{jr} \delta_{ls}$$

then

$$\frac{\partial \lambda_i}{\partial m_{rs}} = - \sum_{j,l} \lambda_i \psi_{ji} \delta_{jr} \delta_{ls} \psi_{li} = - \lambda_i \psi_{ri} \psi_{si} \quad (2-29)$$

$$\text{CASE 3) } r_h = \xi_{rs} : \frac{\partial k_{jl}}{\partial \xi_{rs}} = 0 \quad \text{and} \quad \frac{\partial m_{il}}{\partial \xi_{rs}} = 0$$

then

$$\frac{\partial \lambda_i}{\partial \xi_{rs}} = 0 \quad (2-30)$$

Next consider

$$\frac{\partial \xi_{il}}{\partial r_h}$$

For $r_h = m_{rs}$ or k_{rs}

$$\frac{\partial \xi_{il}}{\partial r_h} = 0 \text{ (neglect) when } i \neq l \quad (2-31)$$

$$\begin{aligned} \frac{\partial \xi_{il}}{\partial r_h} &= \frac{\partial \xi_{ii}}{\partial r_h} = \frac{\partial}{\partial r_h} (2\zeta_i \lambda_i^{1/2} m_{ii}) \text{ when } i = l \\ &= \zeta_i \lambda_i^{-1/2} \frac{\partial \lambda_i}{\partial r_h} m_{ii} \end{aligned} \quad (2-32)$$

where $\frac{\partial \lambda_i}{\partial r_h}$ is defined above (equations 2-28 to 30)

For $r_h = \xi_{rs}$

$$\frac{\partial \xi_{il}}{\partial \xi_{rs}} = \delta_{ir} \delta_{ls} \quad (2-33)$$

Thus the three expressions for $\frac{\partial \tilde{H}_{qm}}{\partial r_h}$ become

CASE 1) $r_h = k_{rs}$

$$\frac{\partial \tilde{H}_{qm}}{\partial k_{rs}} = - \sum_i \tilde{H}_{qi} \psi_{ri} \psi_{si} \tilde{H}_{im} \left(1 + \Omega \zeta_i \lambda_i^{-1/2} m_{ii} \hat{i} \right) \quad (2-34)$$

CASE 2) $r_h = m_{rs}$

$$\frac{\partial \tilde{H}_{qm}}{\partial m_{rs}} = \sum_i \tilde{H}_{qi} \psi_{ri} \psi_{si} \tilde{H}_{im} \lambda_i \left(1 + \Omega \epsilon_i \lambda_i^{-1/2} m_{ii} f \right) \quad (2-35)$$

CASE 3) $r_h = \xi_{rs}$

$$\frac{\partial \tilde{H}_{qm}}{\partial \xi_{rs}} = - \sum_{i,l} \tilde{H}_{qi} \delta_{ri} \delta_{sl} \tilde{H}_{lm} = -\tilde{H}_{qr} \tilde{H}_{sm} \quad (2-36)$$

Finally the evaluation of $\frac{\partial \psi_{nq}}{\partial r_h}$ must be provided

$$\frac{\partial \psi_{ij}}{\partial r_h} = \sum_k \psi_{ik} \frac{\partial \eta_{kj}}{\partial r_h} \quad (2-37)$$

CASE 1) $r_h = k_{rs}$

$$\frac{\partial \eta_{kj}}{\partial k_{rs}} = \sum_{\ell, m} \psi_{\ell k} \left(\frac{1 - \delta_{kj}}{\lambda_j - \lambda_k} \right) \delta_{\ell r} \delta_{ms} \psi_{mj} \quad (2-38)$$

$$\frac{\partial \psi_{ij}}{\partial k_{rs}} = \sum_k \psi_{ik} \psi_{rk} \psi_{sj} \left(\frac{1 - \delta_{kj}}{\lambda_j - \lambda_k} \right) \quad (2-39)$$

* η defined in subsequent equations

CASE 2) $r_h = m_{rs}$

$$\frac{\partial \eta_{kj}}{\partial m_{rs}} = \sum_{\ell, m} \psi_{\ell k} \left[\left(\frac{1 - \delta_{kj}}{\lambda_j - \lambda_k} \right) (-\lambda_j \delta_{\ell r} \delta_{ms}) - \frac{\delta_{kj} \delta_{\ell r} \delta_{ms}}{2} \right] \psi_{mj} \quad (2-40)$$

$$\frac{\partial \psi_{ij}}{\partial m_{rs}} = \sum_k \psi_{ik} \psi_{rk} \psi_{sj} \left(-\frac{\lambda_j (1 - \delta_{kj})}{\lambda_j - \lambda_k} - \frac{\delta_{kj}}{2} \right) \quad (2-41)$$

CASE 3) $r_h = \xi_{rs}$

$$\frac{\partial \psi_{ij}}{\partial \xi_{rs}} = 0 \quad (2-42)$$

In equations 2-16, 21, 25, 26, 33 to 35, 39 to 42, we now have all that is needed to calculate the elements of the sensitivity matrix, $T_{\ell h}$.

2.6 Modification of the Analytic Model (Phase I)

The prior analytic model, as represented by equation 2-8 has the mathematical characteristics shown in equation 2-7. This model, however, does not provide an exact match of the measured test data. If it did, there would be no need to improve it. Therefore there must exist other sets of modes $[\phi]$ and frequencies $[\lambda]$ which will provide a better representation. Therefore our goal is to develop a new analytic model mass matrix, $[m]$; a new analytic model stiffness matrix, $[k]$; and a new generalized damping matrix, $[\xi]$.

These new matrices will not, however, satisfy the requirements of equation 2-7:

$$\begin{aligned} [\phi]^T [{}^oM] [\phi] &\neq [I] \\ [\phi]^T [{}^oK] [\phi] &\neq [\lambda] \end{aligned}$$

The approach to be used to estimate these new matrices is as follows:

- A) Calculate the response of the system at the measurement locations using equation 2-8. $[H(\Omega)]$ is given by equation 2-10 and $[\phi]$, $[\lambda]$, and $[\xi]$ are taken from the unperturbed dynamic model.
- B) Calculate the sensitivity matrix $[T]$ and the "observation" vector. The effective "observation" vector is the difference between the measured responses and the calculated responses (equation 2-2).
- C) Use the minimum variance estimator to provide new mass and stiffness matrices for the analytic system. Input to the estimator consists of the "observation" vector,

the sensitivity matrix, and the previous estimate of the mass and stiffness matrices. The new equations of motion for the "x" system are

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{\tilde{P}\}g(t)$$

- D) Solve for the eigenvalues $[\lambda]$ and eigenvectors $[\psi]$ of the modified "x" system. Normalize these modes such that they have the characteristics:

$$\begin{aligned} [\psi]^T [m] [\psi] &= [I] \text{ diagonal} \\ [\psi]^T [k] [\psi] &= [\lambda] \text{ diagonal} \end{aligned}$$

- E) The revised eigenvectors for the dynamic model, the u system, are given by

$$[\phi] = [{}^o\phi] [\psi]$$

- F) Revise the damping matrix $[\xi]$ (from equation 2-10) to reflect the new eigenvalues while retaining the same damping ratios as were assumed for the initial dynamic model*
 G) Calculate the response using equation 2-8 but with the new eigenvalues $[\lambda]$ and new dynamic model modes $[\phi]$.
 H) Repeat the above steps until convergence is obtained.
 I) Lastly, perform a similar iterative scheme to estimate elements of the damping matrix, $[\xi]$.

*It is assumed here that damping parameters are estimated by a separate operation (see Step 1).

2.7 Sensitivity Matrix Relating Observations and State Vectors for Phase II

The next step in the two-phase system identification procedure being developed is to generate improved mass [M] and stiffness [K] matrices for the dynamic model, the u system. The procedure involved here is similar to that of Phase I, if one considers the "prior" model now to be the dynamic model instead of the analytic model and the "test" data to be the refined parameters of the analytic model instead of the response data.

It is impossible at this point, however, to estimate all of the elements of the [M] and [K] matrices since these matrices may be as large as 750 by 750. Instead these matrices will be broken down into a number of sub-matrices each of which can be scaled individually.

The formulation of Phase II of the system identification procedure is shown below. It envisions that the variable parameters of the dynamic model mass and stiffness matrices will be scaling constants α_h such that

$$\begin{aligned} [{}^oM] &= \text{dynamic model mass matrix} \\ &= [{}^o\bar{M}] + \sum_{h=1}^n \alpha_h [{}^oM]_h \end{aligned} \quad (2-43)$$

and

$[^{\circ}K]$ = dynamic model stiffness matrix

$$= [^{\circ}\bar{K}] + \sum_{h=n+1}^N \alpha_h [^{\circ}K] \quad (2-44)$$

where

- $[^{\circ}\bar{M}]$ = Portion of mass matrix which does not vary,
- $[^{\circ}\bar{K}]$ = Portion of stiffness matrix which does not vary,
- $[^{\circ}M]_h$ = Localized portion of M which is to be scaled,
- $[^{\circ}K]_h$ = Localized portion of K which is to be scaled,
- α_h = Scaling parameter for either $[M]_h$ or $[K]_h$ and
- N = Total number of variable degrees-of-freedom in $[M]$ and $[K]$ combined

In indicial notation equations 2-43 and 2-44 become

$$^{\circ}M_{rs} = ^{\circ}\bar{M}_{rs} + \sum_{h=1}^n \alpha_h ^{\circ}M_{rsh} \quad (2-43)$$

$$^{\circ}K_{rs} = ^{\circ}\bar{K}_{rs} + \sum_{h=n+1}^N \alpha_h ^{\circ}K_{rsh} \quad (2-44)$$

In the Phase II "prior model", it will be assumed that $\alpha_h \equiv 1$ for all h . Variances will be specified independently for each α_h . Along with the revised parameter estimates from the analytical model will be a corresponding covariance matrix, containing the variances and covariances of the Phase II "test data". Thus it follows that for the Phase II parameter estimation, the state vector will be

$$\{r\} = \{\alpha\} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \\ \alpha_N \end{Bmatrix} \quad (2-45)$$

while the first portion of the observation vector will be

$$\{y\}_l = \begin{Bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_l \\ . \\ . \\ . \\ y_n \end{Bmatrix} = \begin{Bmatrix} m_{11} \\ m_{12} \\ . \\ . \\ . \\ m_{jj} \\ m_{jj+1} \\ . \\ . \\ . \\ m_{mm} \end{Bmatrix} \quad \text{for } l = 1 \text{ to } n \quad (2-46)$$

consisting of the diagonal as well as the off-diagonal elements of the revised analytic model mass matrix [m].

The second portion of observation vector is

$$\{y\}_2 = \begin{Bmatrix} k_{11} \\ k_{12} \\ \vdots \\ \vdots \\ y_l \\ \vdots \\ \vdots \\ y_N \end{Bmatrix} = \begin{Bmatrix} k_{11} \\ k_{12} \\ \vdots \\ \vdots \\ k_{jj} \\ k_{jj} + 1 \\ \vdots \\ \vdots \\ k_{mm} \end{Bmatrix} \quad \text{for } l = n+1 \text{ to } N \quad (2-46b)$$

consisting of the diagonal as well as the off-diagonal elements of the revised analytical model stiffness matrix.

At the completion of the Phase I estimation, a non-diagonal generalized mass matrix $[m]$ and a non-diagonal generalized stiffness matrix $[k]$ have been obtained. The objective of the Phase II estimator is to use these two matrices as test data in developing improved dynamic mass and stiffness matrices, $[M]$ and $[K]$. In addition, an improved set of mode shapes $[\phi]$ have resulted from Phase I. These modes can be used to develop a relationship from which the Phase II sensitivity matrix $[T]$ may be obtained.

We have defined

- $[\phi]$ = improved modal matrix of dynamic model
- $[\phi^*]$ = prior modal matrix of dynamic model
- $[\psi]$ = modal matrix of revised analytic system

Thus

$$[\psi]^T [m] [\psi] = [I] \quad (2-47a)$$

$$[\psi]^T [k] [\psi] = [\lambda] \quad (2-47b)$$

What we want to obtain are revised $[M]$ and $[K]$ matrices such that

$$[\phi]^T [M] [\phi] = [\lambda] \quad (2-48a)$$

$$[\phi]^T [K] [\phi] = [I] \quad (2-48b)$$

We also have the relationship between $[\phi]$ and $[\psi]$

$$[\phi] = [{}^o\phi] [\psi] \quad (2-49)$$

First substitute equation 2-49 into equation 2-48b

$$[\psi]^T [{}^o\phi]^T [k] [{}^o\phi] [\psi] = [\lambda] \quad (2-50)$$

From a comparison of this relationship with equation 2-47b, we obtain the relationship

$$[k] = [{}^o\phi]^T [K] [{}^o\phi] \quad (2-51)$$

where $[K]$ is the presently unknown matrix which we wish to generate. Substituting equation 2-44 into the above relationship provides one of the needed relationships.

$$[k] = [{}^o\phi]^T \left([{}^o\bar{K}] + \sum_{h=n+1}^N \alpha_h [{}^oK]_h \right) [{}^o\phi] \quad (2-52)$$

or in indicial notation

$$k_{ij} = \sum_{rs} {}^o\phi_{ri} \left({}^o\bar{K}_{rs} + \sum_{h=n+1}^N \alpha_h {}^oK_{rsh} \right) {}^o\phi_{sj} \quad (2-52)$$

Using equations 2-47a, 48a, and 49 and a similar argument the relationship for $[m]$ may be obtained:

$$[m] = [{}^o\phi]^T \left[[{}^o\bar{M}] + \sum_{h=1}^n \alpha_h [{}^oM]_h \right] [{}^o\phi] \quad (2-53)$$

or

$$m_{ij} = \sum_{r,s} \phi_{ri} \left(\phi_{rs} - \sum_{h=1}^n \alpha_h \phi_{rsh} \right) \phi_{sj} \quad (2-53)$$

The elements of the sensitivity matrix [T] for Phase II parameter estimation are still of the form given by equation 2-19:

$$T_{lh} = \frac{\partial Y_l}{\partial \alpha_h}$$

CASE 1

$$Y_l = m_{ij}$$

$$\begin{aligned} \frac{\partial Y_l}{\partial \alpha_h} &= \sum_{r,s} \phi_{ri} \left(\sum_{\ell=1}^n \phi_{rs\ell} \frac{\partial \alpha_\ell}{\partial \alpha_h} \right) \phi_{sj} \\ &= \sum_{r,s} \phi_{ri} M_{rsh} \phi_{sj} \quad \text{for } \ell = 1 \text{ to } n \quad (2-54) \end{aligned}$$

$$\frac{\partial Y_l}{\partial \alpha_h} = 0 \text{ for } \ell = n+1 \text{ to } N$$

CASE 2

$$Y_l = k_{ij}$$

$$\begin{aligned} \frac{\partial Y_l}{\partial \alpha_h} &= \sum_{r,s} \phi_{rj} \left(\sum_{\ell=n+1}^N \phi_{rs\ell} \frac{\partial \alpha_\ell}{\partial \alpha_h} \right) \phi_{sj} \\ &= \sum_{r,s} \phi_{ri} \phi_{rsh} \phi_{sj} \text{ for } \ell = n+1 \text{ to } N \quad (2-55) \end{aligned}$$

$$\frac{\partial Y_l}{\partial \alpha_h} = 0 \text{ for } \ell = 1 \text{ to } n$$

Notice that the sensitivity matrix is not a function of the scaling parameters, α_n . Only the prior modal matrix $[\phi]$ and the unperturbed dynamic model mass $[M]$ and stiffness $[K]$ matrices enter into its calculation. This, of course, is a direct consequence of the linear relationships between K and k , and between M and m . Because of this linear relationship, the iterative procedure will converge in only one step.

2.8 Modification of the Dynamic Model (Phase II)

The goal of the Phase II estimator is to develop a set of scaling parameters, α , which will improve the mass and stiffness matrices of the dynamic model. It still remains the responsibility of the analyst to select a set of submatrices which when multiplied by scaling factors will improve the model. The methodology described heretofore does not, unfortunately, provide any insight as to how these submatrices should be chosen. The analyst must select submatrices which he thinks might be successful or enlightening based on his experience and knowledge of the structure being investigated. In general, a number of trial configurations may be run before a useful and realistic modified model is obtained.

The basic approach used to estimate the new dynamic model mass and stiffness matrices is as follows:

- A) Complete Phase I to provide "observation" data in the form of generalized mass $[m]$ and stiffness $[k]$ matrices for the dynamic model and a covariance matrix for these elements. It must be emphasized that the generalized mass and stiffness matrices for the dynamic model (the u system) are the mass and stiffness matrices of the analytic model (the x system)

- B) Calculate the sensitivity matrix $[T]$. This need only be done once because the sensitivity matrix is only a function of the unperturbed prior model.
- C) Determine the "calculated" generalized mass and stiffness matrices using the prior modal matrix $[\phi]$ and the unperturbed dynamic model mass and stiffness matrices.
- D) Form the "observation" vector as the difference between the elements of the "calculated" $[m]$ and $[k]$ matrices and the "observation" $[m]$ and $[k]$ matrices.
- E) Use the minimum variance estimator with the observation vector, the sensitivity matrix and the prior estimate of the scaling parameters to provide a new estimate of the scaling parameters. The prior estimate for each scaling parameter is 1.0.

3. EXAMPLE PROBLEMS

At the start of the identification procedure

$$[m] = [\phi]^T [M] [\phi] = [m] = [I] \quad (3-1)$$

$$[k] = [\phi]^T [K] [\phi] = [k] = [\lambda] \quad (3-2)$$

where $[\phi]$ = modes of prior model
 $[M]$ = mass matrix of prior model
 $[K]$ = stiffness matrix of prior model

Ideally, for the analytical examples present in this section, $[m]$ and $[k]$ should converge to

$$[m] = [\phi]^T [M_e] [\phi] \quad (3-3)$$

$$[k] = [\phi]^T [K_e] [\phi] \quad (3-4)$$

where $[M_e]$ and $[K_e]$ are the known (in the real world they will be unknown) mass and stiffness matrices of the correct model. Phase I of the procedure attempts to find the correct $[m]$ and $[k]$ matrices while Phase II uses the results of Phase I to identify $[M_e]$ and $[K_e]$. Only the upper triangular elements of $[m]$ and $[k]$ are sought (i.e., symmetry is assumed).

This section discusses the four example problems investigated to date. Before the individual problems are discussed, a couple of general comments should be noted:

- In the following discussion, "converged" means that the responses calculated with the modified $[m]$ and

[k] are within the specified tolerance band of the exact values.

- "Fit improved" means that [m] and [k] converged to values which significantly improved the prior model but did not fall within the convergence tolerance of the desired values.
- Except for Run 3-3, the "test data" used were exact values obtained from an analytic solution of the exact model. For Run 3-3, random errors with a standard deviation of 5% were introduced into the "Test data".
- A variance, or standard deviation, is specified for each element of [m] and [k] being estimated and for each observation.
- On four of the runs which did not converge, only a portion of the [m] and [k] matrices were being estimated. Neglecting the farthestmost off-diagonal term created difficulty for the 3 degree-of-freedom problem.
- The terms "exact", "test", and "observation" all refer to the same set of parameters. The "exact" model is what we wish to eventually arrive at. It is the model used to artificially generate "test" data from which "observation" points were selected.

The basic conclusion which we draw from these runs is that the estimation methodology works well provided the "prior" model is sufficiently close to "exact" model.

3.1 One Degree-of-Freedom Problem

This, the simplest of all possible problems, has the great advantage that simple relationships can be derived for the sensitivity matrix and a hand solution can be obtained. There is, however, no Phase II estimation required for this problem because $[\phi] \equiv 1$. The equation of motion is

$$M\ddot{u} + C\dot{u} + Ku = R_P \sin \Omega t \quad u \text{ system} \quad (3-5)$$

$$\ddot{u} + 2\zeta\omega_0\dot{u} + \lambda u = \frac{1}{M} R_P \sin \Omega t \quad x \text{ system} \equiv u \text{ system} \quad (3-6)$$

The transfer function is:

$$H(\Omega) = \frac{1/M}{(\omega_0^2 - \Omega^2) + 2\zeta\omega_0\Omega i} = A + B i \quad (3-7)$$

$$A = (\omega_0^2 - \Omega^2) D^{-1}$$

$$B = -2\zeta\Omega\omega_0 D^{-1}$$

where

$$D = M[(\omega_0^2 - \Omega^2)^2 + 4\zeta^2\omega_0^2\Omega^2]$$

$$\lambda = \omega_0^2$$

The sensitivity matrix is given by:

$$T_{lh} = \frac{\partial Y_l}{\partial r_h} = [uu^*]^{-1/2} \left[R_u \frac{\partial R_u}{\partial r_h} + I_u \frac{\partial I_u}{\partial r_h} \right] R_P \quad (3-8)$$

$$T_{lh} = \frac{R_P}{H} \left[A \frac{\partial A}{\partial r_h} + B \frac{\partial B}{\partial r_h} \right] \quad (3-9)$$

where $H = [A^2 + B^2]^{1/2} = \frac{1}{\sqrt{MD}}$

$$P = R_P + I_P \hat{f} = R_P + 0 \hat{f} = R_P$$

First Parameter: $r_h = k$

$$\frac{\partial A}{\partial k} = \frac{[-(\omega_o^2 - \Omega^2)^2 + 4\zeta^2 \Omega^4]}{D^2} \quad (3-10)$$

$$\frac{\partial B}{\partial k} = \frac{[-\zeta \Omega (\omega_o^2 - \Omega^2)^2 + 4\zeta \Omega \omega_o^2 (\omega_o^2 - \omega^2) + 4\zeta^3 \Omega^3 \omega_o^2]}{\omega_o D^2} \quad (3-11)$$

Second Parameter: $r_h = m$

$$\frac{\partial A}{\partial m} = \frac{\Omega^2 (\omega_o^2 - \Omega^2)^2 + 4\zeta^2 \Omega^2 \omega_o^4}{D^2} \quad (3-12)$$

$$\frac{\partial B}{\partial m} = \frac{\zeta \Omega \omega_o}{D^2} \left[(\omega_o^2 - \Omega^2)^2 - 2(\omega_o^2 - \Omega^2) (\omega_o^2 + \Omega^2) + 4\zeta^2 \omega_o^2 \Omega^2 \right] \quad (3-13)$$

The operations of the computer program to calculate the sensitivity matrix elements were verified with this example problem. When the exact model and the prior model were assigned the following values:

● Exact Model:	$K = 1.0$	$\lambda = 1.0$
	$M = 1.0$	$\phi = 1.0$
	$\zeta = 0.1$	

- Prior Model:

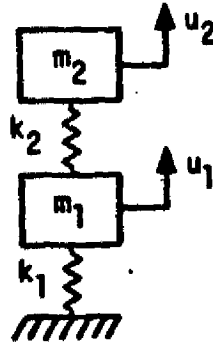
$K = .81$	$\lambda = .81$
$M = 1.0$	$\phi_0 = 1.0$
$\zeta = 0.1$	$\xi = 2\zeta\omega_0 = .18$
$\sigma_k = 1.0$	$\delta_m = 1.0$
$\sigma_s = .18$	

- "Test Data":

l	Ω_l	P_l	u_l	σ_l
1	0.90	1.0	3.821	.052
2	1.10	1.0	3.288	.0375

the estimator converged to within 0.1% of the desired values in 6 iterations.

3.2

Two Degree-of-Freedom Problem

The two degree-of-freedom example is represented by the adjacent sketch. The artificially generated "test" data is based on the following parameters:

$$\begin{array}{ll} k_1 = 1.00 & k_2 = .0125 \\ m_1 = 1.00 & m_2 = .20 \\ \zeta_1 = .10 & \zeta_2 = .05 \end{array}$$

The resulting "exact" frequencies and modes are

$$\begin{aligned} \omega_1 &= 0.1992 \text{ rad/sec} \\ \omega_2 &= 1.0042 \\ [\phi] &= \begin{bmatrix} .02938 & .99957 \\ 2.23510 & -.06570 \end{bmatrix} \end{aligned}$$

The prior model used to investigate the behavior of the Phase I estimator had the following characteristics:

$$\begin{array}{ll} k_1 = 1.00 & k_2 = .008 \\ m_1 = 1.0 & m_2 = 0.2 \\ \zeta_1 = .10 & \zeta_2 = .05 \\ \omega_1 = .1992 & \omega_2 = 1.0042 \end{array}$$

In all cases we restricted the investigation to constant mass and constant damping. The only difference between the "prior" model and the "exact" model was in the stiffness parameters. Eight runs were made. These are summarized below:

Run No.	No. Test Freq.	Range of Test Freq. (rad/sec)	No. Test Points per Frequency	σ^2 for θ [k]	σ^2 for ϕ [m]	Results
1,2	6	.19-1.06	2	.002 .002 .100	.000001 .000001 .000001	Converged to within 5% in 4 iterations.
3	6	.19-1.06	2	.002 .002 .100	.002 .002 .002	Did not improve fit in 8 iterations.
4	10	.10-1.0	1	.002 .002 .100	.002 .002 .002	Converged to within 2% in 4 iterations.
5	10	.10-1.0	1	.002 .100	.002 .002	Only estimated diagonal elements of [m] and [k]. Estimator diverged.
6	10	.10-1.0	1	.002 .002 .100	.00001 .00001 .00001	Converged to within 2% in 4 iterations.
7	10	.10-1.0	1	.002 .100	.00001 .00001	Did not converge to desired model but <u>did</u> improve fit of data.
8	6	.19-1.06	2	.002 .100	.00001 .00001	Only estimated diagonal elements of [m] and [k]. Fit improved.

* Here $\sigma^2 \equiv$ variance. Sequence is 1-1, 1-2, 2-2.

Several typical frequency-response plots are shown in Figures 3, 4, 5, and 6.

On Run 1,2 the estimator was provided with six test points at each node. The tolerance on [k] was set very wide and the tolerance on [m] was set very tight. In 4 iterations the rms error between the calculated and the test responses was reduced from 1.29 to 0.05 with all of the calculated responses within 5 percent of the "test" data. (Figures 3 and 4).

Run 3 is the same as Run 2 except that the prior estimate of the tolerance on [m] was set very wide. As a result the estimator overshoot the desired values and then had trouble finding its way back. After 10 iterations it was not able to improve the fit between the calculated and the test response.

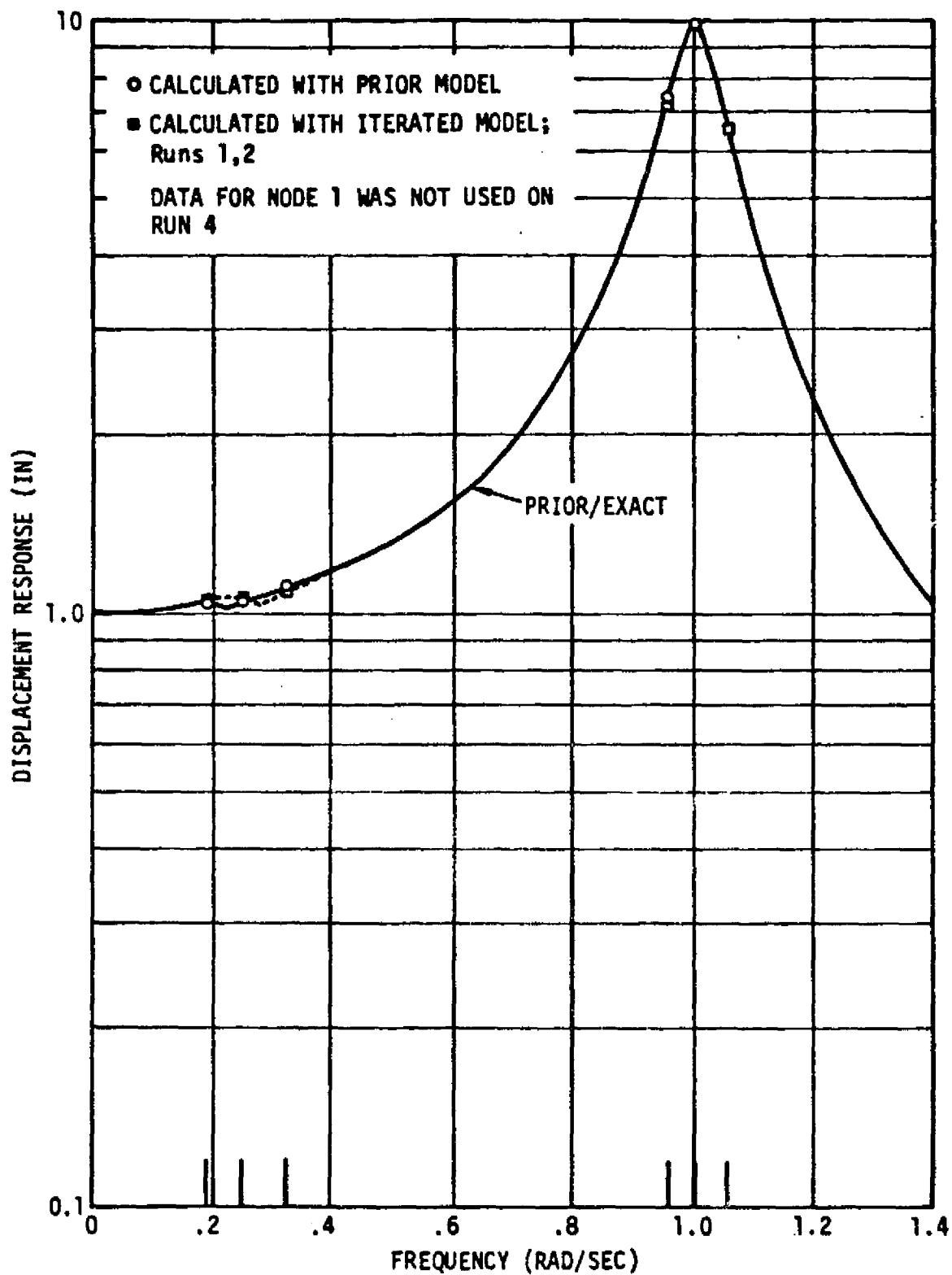


Figure 5. Frequency Response Plots, Two Degree-of-Freedom Example, Node 1

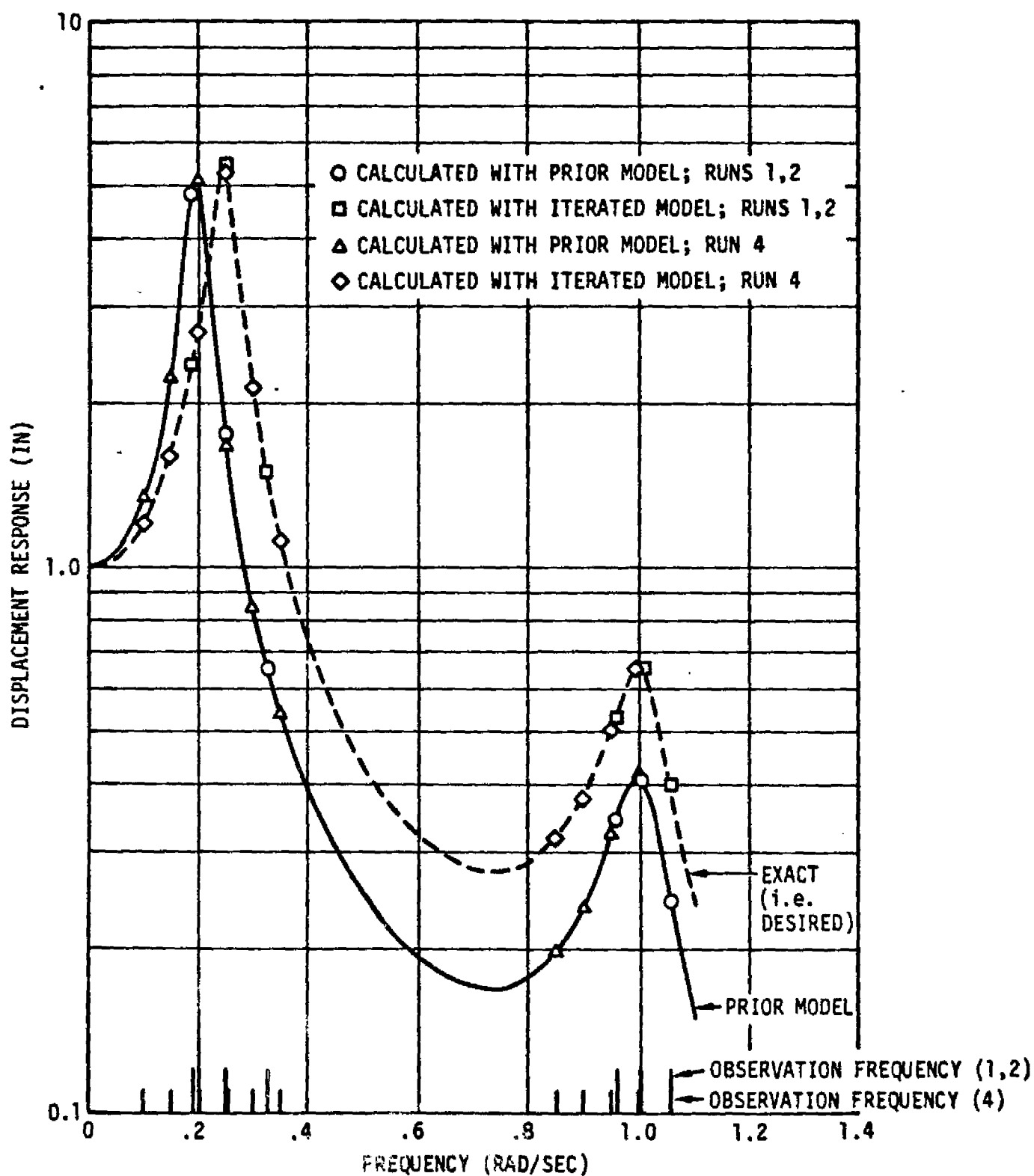


Figure 4. Frequency Response Plots, Two Degree-of-Freedom Example, Node 2

On Run 4 the estimator was only given observation data on one node, node 2. However, the tolerance on both $[m]$ and $[k]$ was left very loose. The estimator was able to match the "test" data to within 2% after only 4 iterations. The $[m]$ and $[k]$ were not as good as on Run 2 however. The new natural frequencies were almost identical to the desired values but the new mode shapes were not as good. Because data was provided for only one node, the new model matched that data very well, but the response of the other node had drifted off. This demonstrates well the fact that a set of parameters which fits the test data well but are unrealistic may be obtained if the test data is not representative of the entire model. (Figures 4 and 5).

On Run 5 we tried to estimate only the diagonal elements of $[m]$ and $[k]$ with wide tolerances on both the prior estimate of $[m]$ and prior estimate of $[k]$. In addition, data was provided only for node 2. These conditions were just too severe. With no coupling between modes and no observation data on node 1, the estimator has no way to estimate the needed mode shapes. After 5 iterations, one of the diagonal elements of the mass matrix went negative.

Run 6 is identical to Run 2 except that observation data was provided at only one node, node 2. The estimator was able to generate a set of parameters that matched the given response data to within 2% after only 4 iterations. But just as on Run 4, the new parameters were not as accurate because the response on the other node was not controlled.

Run 7 is similar to Run 6 with a similar conclusion. After 10 iterations the rms error between observation and calculation was reduced from 1.47 to .02. But because one node was

uncontrolled and because the off-diagonal terms were not estimated, the new model was not realistic even though the given test data was well matched (Figures 5 and 6).

Run 8 is similar to Run 7 with only the diagonal elements being estimated but with data provided at both nodes. The estimator was not able to match the test data as well as on Run 7 or the desired model as well as on Run 2. But a much more realistic model was obtained. The fact that data was provided at both nodes prevented the estimator from pulling away from the desired parameter but the lack of coupling (only diagonal elements were estimated) prevented obtaining a highly accurate set of parameters (Figures 5 and 6).

For execution of Phase II of the parameter estimation, the 2 by 2 matrices were so partitioned that each element was separately estimated:

- Prior Model:

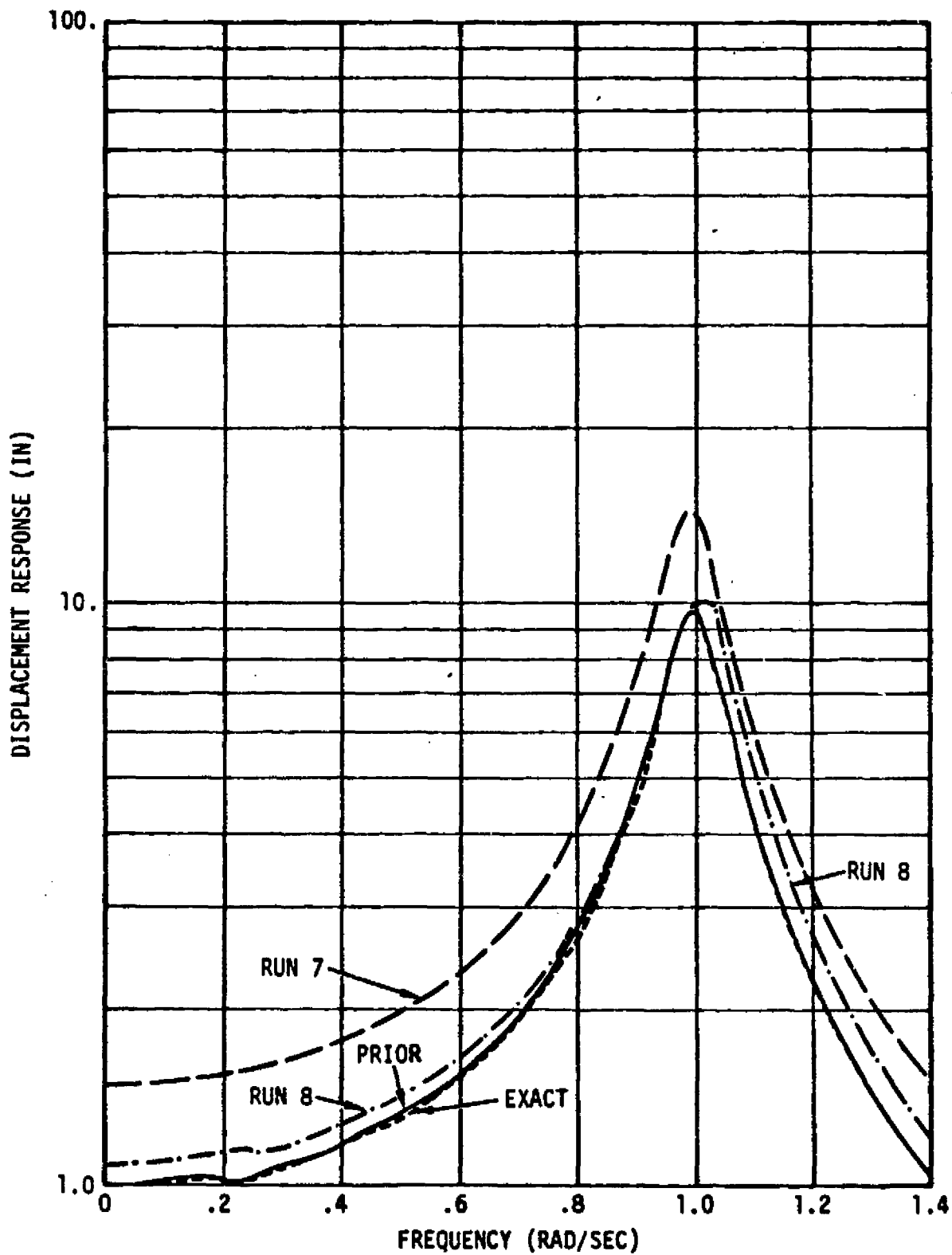
$$[M] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1.0 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 0 & 0.2 \end{bmatrix} = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1.008 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 & -.008 \\ -.008 & .008 \end{bmatrix} = \begin{bmatrix} 1.008 & -.008 \\ -.008 & .008 \end{bmatrix}$$

- Exact Model:

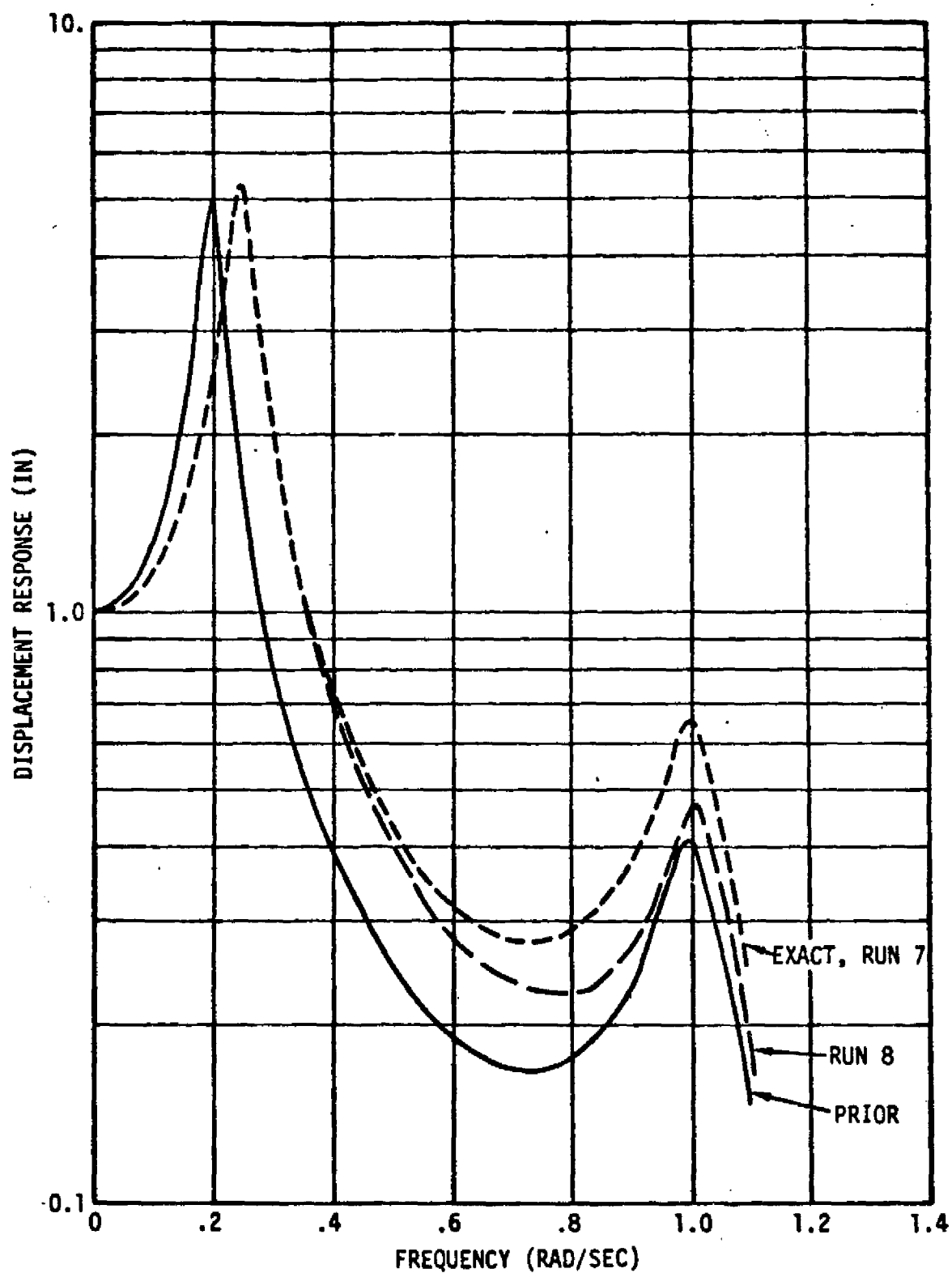
$$[M_e] = \begin{bmatrix} 1.0 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$[K_e] = \begin{bmatrix} 1.0125 & -.0125 \\ -.0125 & .0125 \end{bmatrix}$$



77-1267

Figure 5. Frequency Response Plot, Two Degree of Freedom Example, Node 1



77-1287

Figure 6. Frequency Response Plots, Two Degree of Freedom
Example, Node 2

- Desired Generalized Stiffness Matrix $[\phi]^T [K] [\phi]$:

$$[k] = \begin{bmatrix} .0617917 & -.0103875 \\ -.0103875 & 1.01321 \end{bmatrix}$$

- Prior Modes:

$$[\phi] = \begin{bmatrix} .018470442 & .999829407 \\ 2.23568652 & -.04130120 \end{bmatrix}$$

- Desired Generalized Mass Matrix $[\phi]^T [M] [\phi]$:

$$[m] = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix}$$

- Desired Scaling Parameters

$$\alpha_1 = 1.0$$

$$\alpha_2 = 1.0$$

$$\alpha_3 = 1.004464$$

$$\alpha_4 = 1.562500$$

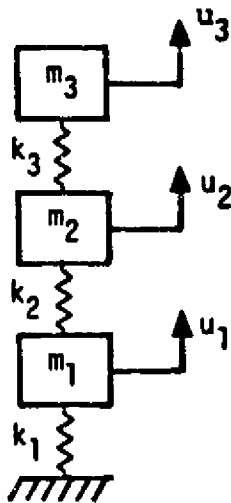
Phase I results from Runs 6, 7, and 8 were all used with the Phase II estimator. In all cases, the estimator produced a set of scaling parameters such that

$$[\phi]^T [M] [\phi] = [m]$$

$$[\phi]^T [K] [\phi] = [k]$$

converged within the specified tolerance. The closeness of the new $[M]$ and $[K]$ matrices to the desired $[M_e]$ and $[K_e]$ matrices depended, of course, on the fidelity with which $[m]$ and $[k]$ had been estimated. Of course, if fewer scaling parameters had been used, an exact match would not have been possible.

3.3

Three Degree-of Freedom-Problem

The three degree-of-freedom example is represented by the adjacent sketch. The artificially generated "test" data is based on the following parameters:

$$k_1 = 47.0, \quad k_2 = 31.0, \quad k_3 = 39.4784$$

$$m_1 = 0.8, \quad m_2 = 1.2, \quad m_3 = 1.0$$

$$\zeta_1 = 0.5, \quad \zeta_2 = .05, \quad \zeta_3 = .05$$

The responses obtained with these parameters are called "test data" "observation" data, or the "exact" model to indicate the various ways this methodology can be approached. The exact natural frequencies are:

$$\omega_1 = 2.700 \text{ rad/sec}$$

$$\omega_2 = 8.284$$

$$\omega_3 = 10.945$$

and the exact modes:

$$[\phi] = \begin{bmatrix} .25436 & .63390 & -.88481 \\ .59217 & .47362 & .50913 \\ .72625 & -.63981 & -.25025 \end{bmatrix}$$

The prior model upon which all of the example-three runs began had the following parameters:

$$\begin{array}{lll}
 k_1 = 39.5 & k_2 = 39.5 & k_3 = 39.5 \\
 m_1 = 1.0 & m_2 = 1.0 & m_3 = 1.0 \\
 \zeta_1 = .05 & \zeta_2 = .05 & \zeta_3 = .05
 \end{array}$$

This yields system natural frequencies of 2.796, 7.835, and 11.322 rad/sec respectively. In this example we have both the mass and the stiffness of the prior model incorrect. Seven runs were made. These are summarized below.

Run No.	No. Test Freq.	Range of Test Freq. (rad/sec)	No. Test Points per Frequency	σ^2 for θ [k]	σ^2 for θ [m]	Results
1	9	2.5-11.3	3	6* 10.0	.10, 5*.01	Converged to within 5% in 9 iterations
2	9	2.5-11.3	3	5* 10.	.10, 4*.01	1-3 term not estimated, estimator diverged.
3	9	2.5-11.3	3	6* 10.0	.10, 5*.01	Fit improved.
4	3	see discussion	3	5* 10.0	.10, 4*.01	Estimator diverged
5	3	see discussion	3	6* 10.0	.10, 5*.01	Estimator diverged.
5a	3	see discussion	3	6* 10.0	.10, 5*.01	Fit improved.
5b	3	see discussion	3	6* 10.0	.10, 5*.01	Fit improved.

* here $\sigma^2 \equiv$ variance. Sequence is 1-1, 1-2, 1-3, 2-2, 2-3, and 3-3. N* means repeat N times.

On Run 1 the estimator was provided with 9 test points at all three modes. These test points bracketed each exact/prior natural frequency. The tolerance on both [m] and [k] was set wide. In 9 iterations, the rms error between calculated and exact response was reduced from .040 to .002 and a good approximation of the desired [m] and [k] matrices was obtained.

Run 2 is identical to Run 1 except only the diagonal and first off-diagonal terms of $[m]$ and $[k]$ were estimated. Just as happened in a similar situation on example two, the estimator overshot on its first iteration so far that it was not able to find its way back.

Run 3 is identical to Run 1 except for a new wrinkle: random error was introduced into the test data. In this instance, the estimator was not able to bring all of the calculated responses within 5% of the test values but it was able to converge on to a set of parameters which significantly improved the comparison. The rms error was reduced from .040 to .007.

Run 4 presented a very severe problem to the estimator because only three observation frequencies were used: one at each of the natural frequencies of the exact (i.e., the test) model. The sensitivity matrix $[T]$ is closely related to the transfer function matrix $[H]$. When the observations do not bracket the natural frequencies of the prior model, the sensitivity of the estimator is somewhat unpredictable. In this case, the estimator overshot so far that the diagonal elements of the estimated matrices went negative. The number and location of the observation points are very important in determining whether or not a successful run will be accomplished. Run 5 was made to see whether addition of the 1-3 term to the estimation would have any beneficial effect. It didn't.

Following Runs 4 and 5 of this problem, a two-way step retarder was incorporated into the methodology. This retarder checked whether or not the new estimate had a negative on the diagonal, or an increased rms error. If either was found, it halved the step and tried again. Run 5 was then repeated as Run 5a. The

lack of data still presented a serious obstacle. But after 20 iterations or partial steps, the rms error was reduced from .21 to .011. Then the retarder was changed from a 1/2 to a 1/4 step and Run 5 repeated again. Again progress was slow but positive. After 27 steps the rms error was reduced from .021 to .006.

The [m] and [k] matrices generated on Run 1 were used as input to the Phase II estimator. Sufficient scaling parameters were used to provide the opportunity for an exact match. A close fit was obtained after only a couple of iterations (see below).

Prior Model:

$$\begin{aligned}
 [m] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1.0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 1.0 & & \\ & 1.0 & \\ & & 1.0 \end{bmatrix} \\
 [k] &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 39.4784176 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &\quad + \alpha_4 \begin{bmatrix} 39.4784176 & -39.4784176 & 0 \\ -39.4784176 & 39.4784176 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &\quad + \alpha_5 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 39.4784176 & -39.4784176 \\ 0 & -39.4784176 & 39.4784176 \end{bmatrix}
 \end{aligned}$$

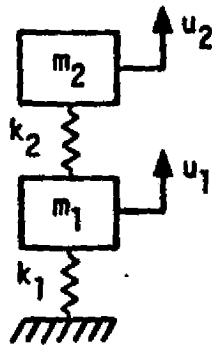
Exact Model:

$$[M_e] = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.0 \end{bmatrix}$$

$$[K_e] = \begin{bmatrix} 78.0 & -31.0 & 0 \\ & 70.47841761 & -39.47841761 \\ \text{SYMMETRIC} & & 39.47841761 \end{bmatrix}$$

<u>Scaling Parameter</u>	<u>Desired</u>	<u>Obtained</u>
α_1	0.8000	0.8017
α_2	1.2000	1.1809
α_3	1.1905	1.2039
α_4	.7852	.7664
α_5	1.0000	.9934

3.4

Two Degree-of-Freedom Problem: Close Modes

To investigate the effects of close modes on the estimator performance, the two degree-of-freedom model was adjusted to yield two resonant frequencies within 20% of each other (Figures 7 and 8). The artificially generated "test" data is based on the following parameters:

$$\begin{aligned} k_1 &= 33.333 & k_2 &= 1.0 \\ m_1 &= 28.83 & m_2 &= .8649 \\ \zeta_1 &= .05 & \zeta_2 &= .10 \end{aligned}$$

The resulting frequencies and modes are:

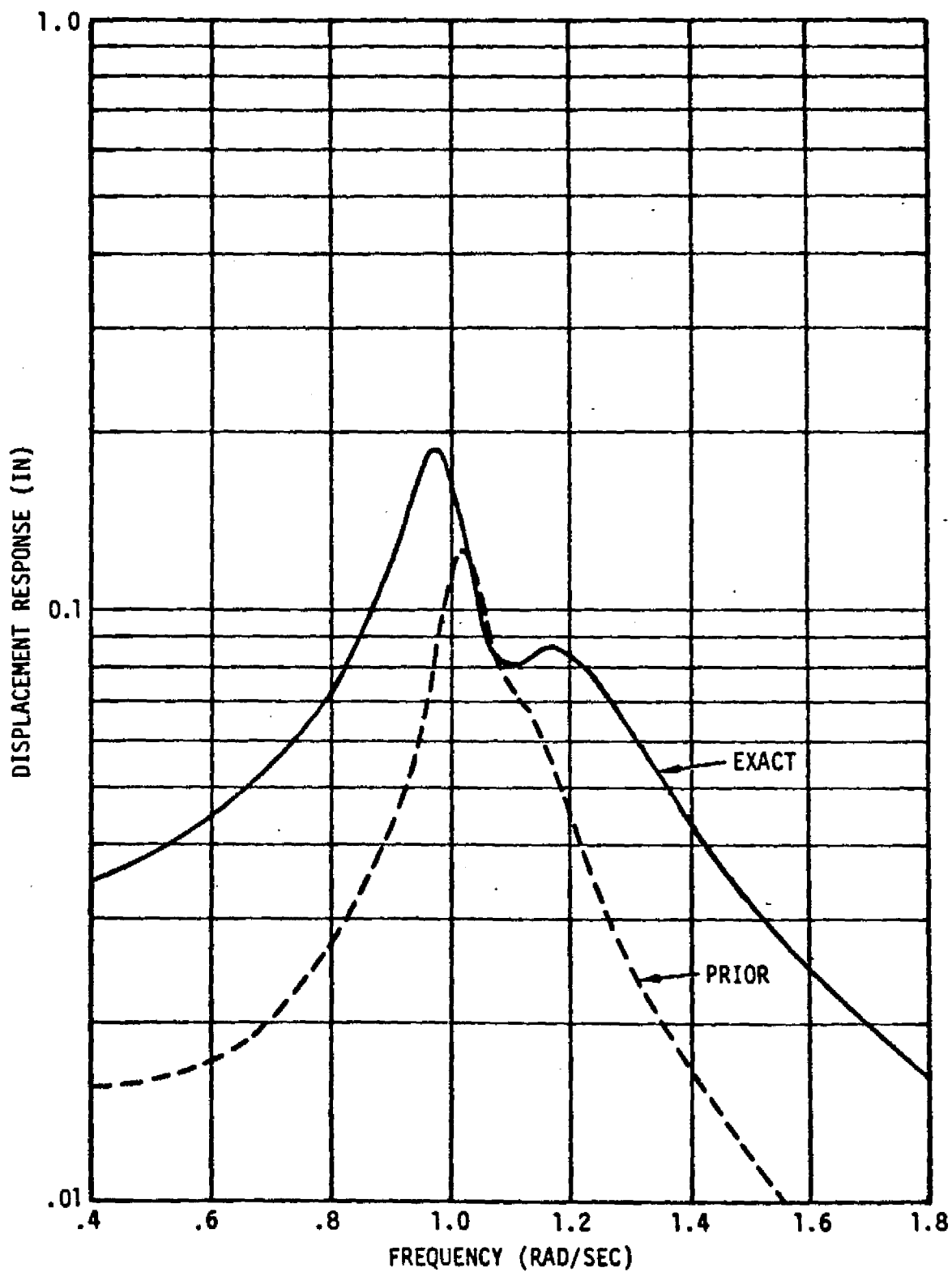
$$\begin{aligned} \omega_1 &= .98617 \text{ rad/sec} \\ \omega_2 &= 1.17214 \end{aligned}$$

$$[\phi] = \begin{bmatrix} .12588 & .13726 \\ .79246 & -.726767 \end{bmatrix}$$

The sinusoidal forcing function was applied at node 1 with a magnitude of 1 pound.

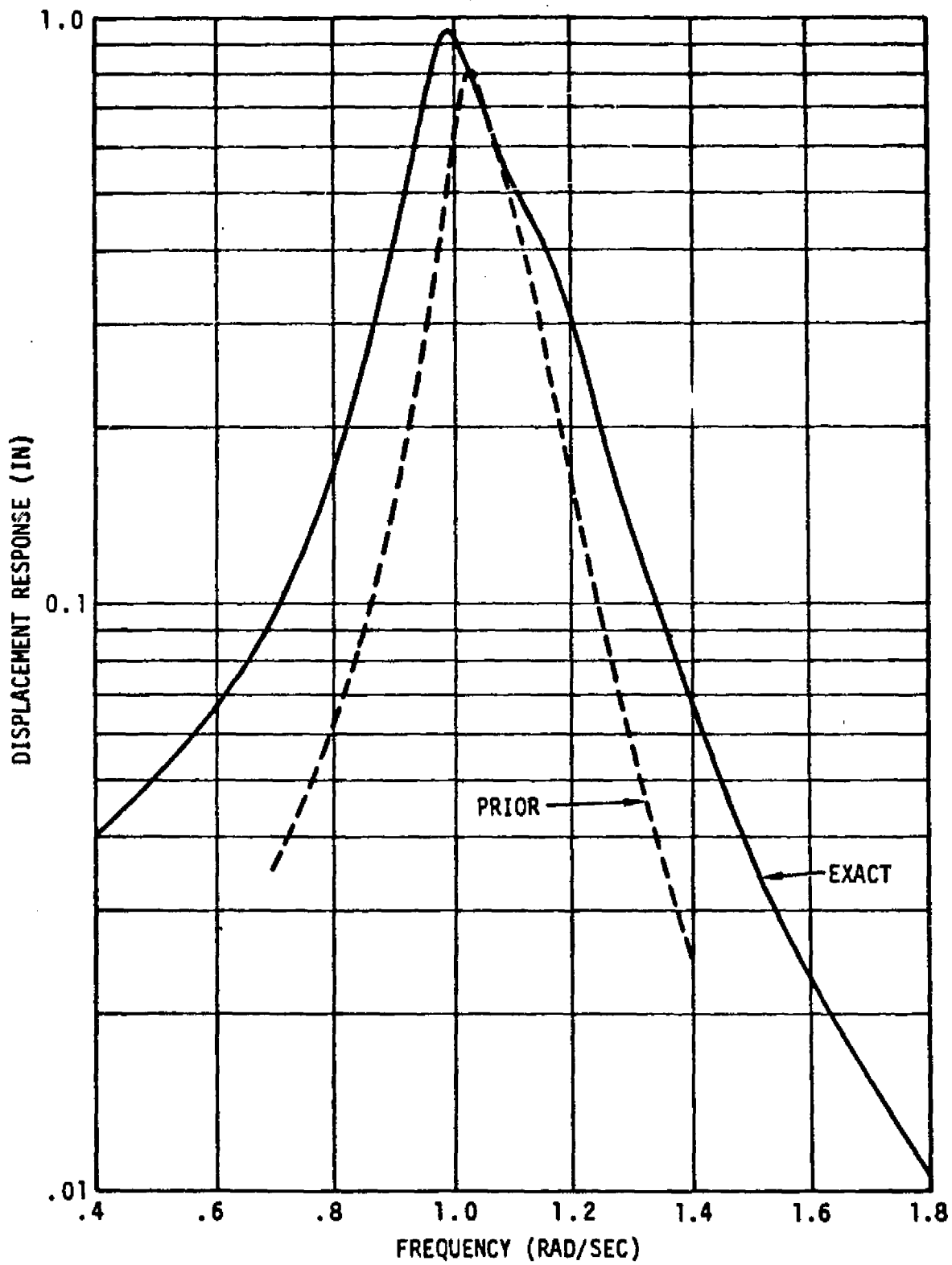
The prior model used here had these characteristics:

$$\begin{aligned} k_1 &= 86.715 & k_2 &= 1.00 \\ m_1 &= 75.00 & m_2 &= .8649 \\ \zeta_1 &= .05 & \zeta_2 &= .10 \end{aligned}$$



17-1287

Figure 7. Frequency Response Plots, Example 4 - Close Modes, Node 1



77-1287

Figure 8. Frequency Response Plots, Example 4 - Close Modes, Node 2

which yielded natural frequencies of 1.0191 and 1.1346 rad/sec.

Three runs were made:

Run No.	No. Test Freq.	Range of Test Freq. (rad/sec)	No. Test Points per Frequency	σ^2 for [k]	σ^2 for [m]	Results
1	10	.9-1.25	2	all .0025	all .0025	Unsuccessful - no step retarder.
2	10	.9-1.25	2	all .0025	all .0025	Unsuccessful - one-way step retarder.
3	10	.9-1.25	2	all .0025	all .0025	See discussion.

The first run was performed without the step retarder discussed in the previous example. As a result the second iteration produced such a large change in [m] that a negative diagonal element was generated. Next a one-way step retarder was added to prevent large increases in the rms error between the calculated and the measured (i.e., test or exact) responses. This enabled the estimator to have a better first iteration but one which still overshoot the mark. Thus it still produced a negative diagonal mass element when it tried to return (Run 2). Next the two-way step retarder was added. This prevented the estimator from "blowing-up" but it did not enable it to effectively improve the model. Apparently, the prior model is so far off in its mass and stiffness that the sensitivity matrix elements are not representative of the true model. This is true even though the frequencies are well matched.

Phase II of the estimation procedure was not performed on this example.

4. METHOD FOR IMPROVING PRIOR ANALYTIC MODEL

One of the difficulties pointed up in the foregoing examples is that when the prior analytic model is not sufficiently close to the physical system, in the sense that computed frequency response does not match experimentally measured frequency response well enough, then the estimation procedure may not converge properly. Another difficulty pointed up is that of identifying which of the off-diagonal elements of $[m]$ and $[k]$ need to be included in the parameter vector being estimated, i.e. which elements of these two matrices need to be adjusted to achieve a proper fit of the model to the data. This section documents an analysis aimed at resolving these difficulties.

4.1 First Order Correction

Two observations may be made at the outset: (1) there are certain measurement data available from modal survey tests which are not explicitly used in the estimation procedure described in Section 2, namely mode shapes and frequencies; and (2) previous experience with MOUSE [1] indicates that when such information is used in a linear estimator to revise a prior model, most of the change to the parameters of the prior model is made in the first step of the iterative estimation procedure. These observations suggest that direct use of measured modal data may provide the key to conditioning (or improving) the prior analytical model so as to enable it to converge to the measured response data using the methods of Section 2. The implementation of some procedure to affect this conditioning, in conjunction with the existing computer programs, has the potential capability of combining the distinct advantages of two different approaches.

The first approach utilizes explicit modal data (where available) as a basis for improving the analytical model; the second approach utilizes sinusoidal response data where "pure modes" are not required. The primary advantage of the first approach is that where good modal resolution has been achieved, numerical computation appears to be more stable. However, only mass and stiffness parameters can be estimated, no damping. The advantages of the second approach are (1) damping parameters can be estimated as well as mass and stiffness, and (2) "pure modes" are not required. Thus, in the case of closely spaced modes where experimental isolation of the modes may be hard to achieve, sinusoidal response data reflecting multi-mode participation can be used.

Let us assume that our "prior dynamic model" is given in terms of the mass and stiffness matrices $[^{\circ}M]$ and $[^{\circ}K]$. Here again, the left-hand superscript notation is used to avoid double subscripting later on. The corresponding eigenvalues, $^{\circ}\lambda_j$, and eigenvectors, $\{^{\circ}\phi\}_j$, are obtained by solving the eigenproblem

$$([^{\circ}K] - ^{\circ}\lambda_j[^{\circ}M])\{^{\circ}\phi\}_j = \{0\} \quad (4-1)$$

Now suppose that there exists a "true dynamic model" in the sense that it produces a set of eigenvalues, λ_j , and eigenvectors, $\{\phi\}_j$, which agree with measured test data. Let that model be represented by the mass and stiffness matrices $[M]$ and $[K]$ respectively. Thus

$$([K] - \lambda_j[M])\{\phi\}_j = \{0\} \quad (4-2)$$

We can relate the parameters and modal characteristics of the prior model to those of the "true" model by

$$[K] = [^{\circ}K] + [\Delta K] \quad (4-3a)$$

$$[M] = [^{\circ}M] + [\Delta M] \quad (4-3b)$$

$$\lambda_j = ^{\circ}\lambda_j + \Delta\lambda_j \quad (4-3c)$$

$$\{\phi\}_j = \{^{\circ}\phi\}_j + \{\Delta\phi\}_j \quad (4-3d)$$

Then, substitution of equations 4-3 into equation 4-2 gives

$$\left[(^{\circ}K + \Delta K) - (^{\circ}\lambda_j + \Delta\lambda_j)(^{\circ}M + \Delta M) \right] \{^{\circ}\phi_j + \Delta\phi_j\} = \{0\} \quad (4-4)$$

Subtraction of equation 4-1 from 4-4 leads to the first-order approximation:

$$[^{\circ}K - ^{\circ}\lambda_j ^{\circ}M] \{\Delta\phi\}_j + [\Delta K - ^{\circ}\lambda_j \Delta M - \Delta\lambda_j M] \{^{\circ}\phi\}_j = \{0\} \quad (4-5)$$

Following the same procedures used in deriving eigenvalue and eigenvector derivatives [3], we first premultiply equation 4-5 by $\{^{\circ}\phi\}_i^T$. When $i = j$, we find that

$$\Delta\lambda_j = \{^{\circ}\phi\}_j^T [\Delta K - ^{\circ}\lambda_j \Delta M] \{^{\circ}\phi\}_j \quad (4-6)$$

when $i \neq j$, we find that

$$\{^{\circ}\phi\}_i^T \left([^{\circ}K - ^{\circ}\lambda_j ^{\circ}M] \{\Delta\phi\}_j + [\Delta K - ^{\circ}\lambda_j \Delta M] \{^{\circ}\phi\}_j \right) = \{0\}$$

The key step at this point is to recognize that $\{\Delta\phi\}$ may be expressed as a linear combination of the prior model mode shapes $\{^{\circ}\phi\}$. Thus:

$$\{\Delta\phi\}_j \equiv \{^{\circ}\phi\} \{\Delta\eta\}_j \quad (4-7)$$

Substitution of this form into the previous equation results in the following for $i \neq j$:

$$\Delta\eta_{ij} = (\circ\lambda_j - \circ\lambda_i)^{-1} \{\circ\phi\}_i^T [\Delta K - \circ\lambda_j \Delta M] \{\circ\phi\}_j \quad (4-8)$$

Note the similarity in form between equation 4-6 and equation 4-8.

One additional piece of information is required to complete the picture, so to speak, i.e. we need a corresponding expression for $\Delta\eta_{ij}$, the diagonal elements of the matrix $[\Delta\eta]$. Clearly equation 4-8 is meaningless whenever $i = j$. The additional piece of information is provided by the normalization condition imposed on the eigenvectors $\{\phi\}_j$. Let us require that

$$\{\phi\}_j^T [M] \{\phi\}_j \equiv 1 \quad (4-9)$$

Since we have assumed originally that

$$\{\circ\phi\}_j^T [\circ M] \{\circ\phi\}_j \equiv 1$$

we find that substitution of equation 4-3 and 4-7 into 4-9 yields the final equation

$$\Delta\eta_{jj} = -\frac{1}{2} \{\circ\phi\}_j^T [\Delta M] \{\circ\phi\}_j \quad (4-10)$$

At this time we recall that our objective is to improve our "prior analytic model, "originally specified in terms of reduced mass and stiffness matrices $[\circ m]$ and $[\circ k]$. Let us define a first-order correction to these matrices in terms of the incremental adjustments to its elements

$$\{\circ\phi\}_i^T [\Delta M] \{\circ\phi\}_j \equiv \Delta m_{ij} \quad (4-11a)$$

$$\{\circ\phi\}_i^T [\Delta K] \{\circ\phi\}_j \equiv \Delta k_{ij} \quad (4-11b)$$

We further recognize the symmetric form of $[m]$ and $[k]$ in general, so that

$$\Delta m_{ij} = \Delta m_{ji}$$

$$\Delta k_{ij} = \Delta k_{ji}$$

Thus, if we are to determine an improved prior analytic model in terms of say $[\hat{m}]$ and $[\hat{k}]$, we find that we may use

$$[\hat{m}] = [{}^{\circ}m] + [\Delta m] = [I] + [\Delta m] \quad (4-12a)$$

$$[\hat{k}] = [{}^{\circ}k] + [\Delta k] = [{}^{\circ}\lambda] + [\Delta k] \quad (4-12b)$$

In general, if $[m]$ and $[k]$ are of order $n \times n$, we will require $(n^2 + n)/2$ terms to define each of $[\Delta m]$ and $[\Delta k]$, or a total of $(n^2 + n)$ terms for both of them. With reference to equations 4-6 and 4-8, and 4-10, we find that if we know $\Delta \eta_{ij}$ for all i and j , each ranging from one to n , and in addition knew $\Delta \lambda_j$ for all j , then we can solve these three equations directly for Δm_{ij} and Δk_{ij} .

To complete the derivation, we recall from equation 2-23 that

$$[{}^{\circ}\phi]\{\psi\}_j = \{\phi\}_j$$

and recognize from equation 4-7 that

$$[\Delta \eta] \equiv [\psi] - [I] \quad (4-13)$$

Finally, we recognize that the cross-orthogonality between the measured "test modes" $\{\hat{\phi}\}_j$ and the original modes $\{{}^{\circ}\phi\}_i$ may be expressed in terms of

$$[{}^{\circ}\phi]^T [{}^{\circ}M] [\hat{\phi}] = [{}^{\circ}\phi]^T [{}^{\circ}M] [{}^{\circ}\phi] [\hat{\psi}] = [\hat{\psi}] \quad (4-14)$$

Therefore,

$$\Delta\eta_{ij} \equiv \hat{\psi}_{ij} - \delta_{ij} \quad (4-15)$$

where δ_{ij} is the Kronecher delta.

In summary, given the test frequencies, $\hat{\omega}_j$, we may compute "test eigenvalues" $\hat{\lambda}_j = \hat{\omega}_j^2$. Given the test mode shapes, $\hat{\phi}_j$, we may compute the cross-orthogonality coefficients $\hat{\psi}_{ij}$. From this information we can then compute

$$\begin{aligned} \Delta\lambda_j &= \hat{\lambda}_j - {}^o\lambda_j \\ \Delta\eta_{ij} &= \hat{\psi}_{ij} - \delta_{ij} \end{aligned}$$

Using equations 4-6, 4-8 and 4-10 we may procede to compute Δm_{ij} and Δk_{ij} . Then from 4-12 we obtain the mass and stiffness matrices, $[\hat{m}]$ and $[\hat{k}]$, for an improved prior analytical model.

The computations involving equations 4-6, 4-8 and 4-10 require only the solution of pairs of linear algebraic equations. To obtain the diagonal terms, equations 4-6 and 4-10 are used, recognizing that

$$\{\phi\}_j^T [\Delta M] \{\phi\}_j = \Delta m_{jj}$$

$$\{\phi\}_j^T [\Delta K] \{\phi\}_j = \Delta k_{jj}$$

Thus 4-6 and 4-10 become

$$\Delta\lambda_j = \Delta k_{ij} - {}^o\lambda_j \Delta m_{ij} \quad (4-16a)$$

$$\Delta\eta_{ij} = \hat{\psi}_{ij} - 1 = \frac{1}{2} \Delta m_{ij} \quad (4-16b)$$

In a similar manner, equation 4-8 yields

$$\Delta \eta_{ij} = (\Delta k_{ij} - \lambda_j \Delta m_{ij}) / (\circ \lambda_j - \circ \lambda_j) \quad (4-17a)$$

$$\Delta \eta_{ji} = (\Delta k_{ij} - \lambda_i \Delta m_{ij}) / (\circ \lambda_i - \circ \lambda_j) \quad (4-17b)$$

4.2 Criteria for Banding [m] and [k]

It may be recalled that the original (prior) "dynamic model" was defined by $[\circ M]$ and $[\circ K]$ with corresponding equations of motion

$$[\circ M]\{\ddot{u}\} + [\circ C]\{\dot{u}\} + [\circ K]\{u\} = \{f(t)\}$$

Transformation to a modal coordinate system was defined such that $\{u\} = [\circ \phi]\{x\}$, resulting in what we have called the prior "analytic model." Equations of motion in the x-coordinate system are

$$[\circ m]\{\ddot{x}\} + [\circ c]\{\dot{x}\} + [\circ k]\{x\} = [\circ \phi]^T\{f(t)\}$$

We have denoted the (unknown) "true" values of mass, damping and stiffness matrices in these two coordinate systems as

$[M], [C], [K]$ Dynamic Model

$[m], [c], [k]$ Analytic Model

where

$$[\circ \phi]^T [M] [\circ \phi] \equiv [m]$$

$$[\circ \phi]^T [C] [\circ \phi] \equiv [c]$$

$$[\circ \phi]^T [K] [\circ \phi] \equiv [k]$$

We have also derived in Section 4.1, approximate values for $[m]$ and $[k]$ which are denoted by $[\hat{m}]$ and $[\hat{k}]$ respectively.

Let us now consider the following equations of motion

$$\begin{aligned} [\hat{m}]\{\ddot{x}\} + [\hat{c}]\{\dot{x}\} + [\hat{k}]\{x\} &= [{}^o\phi]^T\{f(t)\} \\ &= [{}^o\phi]^T\{P\}g(t) \end{aligned} \quad (4-18)$$

We have not yet defined $[\hat{c}]$, but introduce the notation for convenience here. Transformation to the frequency domain gives

$$([\hat{k}] - \Omega^2[\hat{m}] + i\Omega[\hat{c}])X(i\Omega) = [{}^o\phi]^T\{P\}G(i\Omega) = \{\tilde{P}\}G(i\Omega) \quad (4-19)$$

It is furthermore convenient to introduce the notation

$$\bar{X}(i\Omega) = X(i\Omega)/G(i\Omega) \quad (4-20)$$

Now let us recall that

$$[\hat{m}] = [I] + [\Delta m] \quad (4-21a)$$

$$[\hat{k}] = [{}^o\lambda] + [\Delta k] \quad (4-21b)$$

and for the sake of consistency, let

$$[\hat{c}] = [{}^o c] + [\Delta c] \quad (4-21c)$$

where $[{}^o c]$ is diagonal such that

$${}^o c_{jj} = 2{}^o \zeta_j \sqrt{{}^o \lambda_j}$$

and ${}^o \zeta_j$ is the prior estimate of the critical damping ratio for the j^{th} mode.

It follows that we may define the dynamic impedance matrix in the x -coordinate system as

$$[\tilde{Z}(\hat{\Omega})] = [\hat{k}] - \Omega^2[\hat{m}] + i\Omega[\hat{c}] \quad (4-22)$$

where the tilde over the Z is used to denote the fact that elements of $\tilde{Z}$ correspond to the x-coordinate system.*

Recognizing the form of equations 4-21, we may express $[\tilde{Z}(\hat{\Omega})]$ as the sum of diagonal and nondiagonal complex matrices.

$$[\tilde{Z}(\hat{\Omega})] = [{}^{\circ}\tilde{Z}(\hat{\Omega})] + [\Delta\tilde{Z}(\hat{\Omega})] \quad (4-23)$$

where

$$[{}^{\circ}\tilde{Z}(\hat{\Omega})] = [{}^{\circ}\lambda] - \Omega^2[I] + i\Omega[{}^{\circ}c]$$

$$[\Delta\tilde{Z}(\hat{\Omega})] = [\Delta k] - \Omega^2[\Delta m] + i\Omega[\Delta c]$$

We can then use $[{}^{\circ}\tilde{Z}(\hat{\Omega})]$ to generate a complex scaling transformation,

$$[D(\hat{\Omega})] \equiv [{}^{\circ}\tilde{Z}(\hat{\Omega})]^{-1/2} \quad (4-24)$$

such that a new coordinate system, γ , is defined by $\{x\} = [D]\{\gamma\}$. Then equation 4-19 becomes**

$$([I] + [D][\Delta\tilde{Z}][D])\{\Gamma\} = [D]\{\tilde{P}\} \quad (4-25)$$

The purpose of making this scaling transformation is, that now the numerical significance of Δm_{ij} and Δk_{ij} may be examined, by evaluating the corresponding term of the matrix

*To be entirely consistent in notation, the carat (^) should appear over the $\tilde{Z}$, but is omitted for convenience.

**The $\tilde{\Gamma}$ notation in equation 4-25 is used in the same sense as the $\tilde{X}$ in equation 4-20.

$[D][\Delta \tilde{Z}][D]$ in equation 4-25. If any term of this matrix is small compared to unity, the corresponding Δm_{ij} , Δk_{ij} and Δc_{ij} may be neglected regardless of its own particular magnitude.

Since we are particularly interested in the significance of only $[\Delta m]$ and $[\Delta k]$ for the time being, we may examine only the real part, $R[\Delta \tilde{Z}]$, of $[\Delta \tilde{Z}]$ initially.

We may reason that a suitable criterion for neglecting Δm_{ij} and Δk_{ij} is that

$$| \{e\}_i^T [D] ([\Delta k] - \Omega [\Delta m]) [D] \{e\}_j | < \epsilon \quad (4-26)$$

where e_j denotes the j^{th} column of the identity matrix, $[I]$, and $\epsilon \ll 1$. That is, if we define

$$[\Delta \tilde{Z}^R] \equiv [D] (R[\Delta \tilde{Z}]) [D],$$

equation 4-26 may be stated

$$| \Delta \tilde{Z}_{ij}^R | < \epsilon$$

Following similar logic to the derivation shown in Reference [4], we focus our attention on the resonant frequencies where we assume that $\Omega = \omega_i < \omega_j$; $\omega_j \equiv \sqrt{\omega_i \lambda_j}$. The criterion for neglecting Δm_{ij} and Δk_{ij} is thus found to be

$$\left| \Delta k_{ij}/\omega_i^2 - \Delta m_{ij} \right| \left\{ \frac{1/(2\zeta_i)}{\left[\left(\frac{\omega_j^2}{\omega_i^2} - 1 \right)^2 + 4\zeta_j^2 \frac{\omega_j^2}{\omega_i^2} \right]^{1/2}} \right\}^{\frac{1}{2}} < \epsilon \quad (4-27)$$

To simplify the notation, we may define

$$\omega_j/\omega_i = \beta > 1 \quad (\text{frequency ratio})$$

$$1/2\zeta_i = Q_i \quad (\text{dynamic amplification})$$

Then, for light damping ($\zeta_j \ll 1$) whenever $(\beta^2 - 1)^2 \gg \beta^2/Q_j^2$, equation 4-27 reduces to

$$\left| \Delta k_{ij}/\omega_i^2 - \Delta m_{ij} \right| \sqrt{\frac{Q_i}{(\beta^2 - 1)}} < \epsilon \quad (4-28)$$

As a practical matter, we might expect a value of $\epsilon = 0.2$ to work.

It is of interest to note from Reference [4] that the companion criterion for neglecting Δc_{ij} is

$$\left| \Delta c_{ij}/\omega_i \right| \sqrt{\frac{Q_i}{(\beta^2 - 1)}} < \epsilon \quad (4-29)$$

To conclude this section, we recall that our original objective was to band the matrices [m] and [k] for computational purposes. Since the rows and columns of these matrices are ordered by frequency, it is evident that the frequency ratio, $\beta = \omega_j/\omega_i$, which appears in the denominator of equation 4-28, acts to attenuate the importance of Δm_{ij} and Δk_{ij} as one gets further away from the diagonal.

4.3 Sample Calculations

The banding criterion derived in the previous section may be applied to some of the examples worked in Section 3. In particular, there were two runs made with banded [m] and [k] matrices. One converged and the other did not.

The first example corresponds to Example 2, Run 8, of Section 3.2. This run did converge in the sense that numerical computations stabilized after achieving a significantly improved fit of the data. The example consists of a two degree-of-freedom system with 10% damping in each mode, where

$$[k] = \begin{bmatrix} .0618 & -.0104 \\ -.0104 & 1.0132 \end{bmatrix}$$

$$[m] = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

Since $[^{\circ}k]$ and $[^{\circ}m]$ are both diagonal (i.e. off-diagonal elements are identically zero) in the prior model, it follows that

$$\Delta k_{12} = -.0104$$

$$\Delta m_{12} = 0$$

It is also given in this example that

$$\left. \begin{array}{l} \omega_1^2 = .0618 \\ \omega_2^2 = 1.0132 \end{array} \right\} \beta^2 = 16.39$$

corresponding to the diagonal terms of $[k]$, and

$$\zeta_1 = 0.1 \quad + \quad Q_1 = 5.0$$

We wish to determine whether in fact, equation 4-29 is satisfied.

$$\left| \frac{\Delta k_{12}}{\omega_1^2} - \Delta m_{12} \right| \sqrt{\frac{Q_1}{(\beta^2 - 1)}} = .097$$

which is less than the tentative value of $\epsilon = 0.2$ suggested earlier, i.e. the criterion worked.

The second example corresponds to Example 3, Run 2, of Section 3.3. In this case, numerical computations did not converge when the (1,3) elements of [m] and [k] were excluded from the set of parameters being estimated. The following data from Example 3 apply:

$$\Delta k_{13} = -3.35$$

$$\Delta m_{13} = -.136$$

$$\left. \begin{array}{l} \omega_1^2 = 7.35 \\ \omega_3^2 = 118.5 \end{array} \right\} \beta^2 = 16.12$$

$$\zeta_1 = 0.05 \quad + \quad Q_1 = 10.0$$

We wish to determine again whether equation 4-28 is satisfied.

$$\left| \frac{\Delta k_{13}}{\omega_1^2} - \Delta m_{13} \right| \sqrt{\frac{Q_1}{(\beta^2 - 1)}} = 0.481$$

which is greater than the tentative value of $\epsilon = 0.2$ suggested on Section 4.2, i.e. the criterion worked again.

The suggested value of $\epsilon = 0.2$ is only a first guess at establishing a working criterion for excluding off-diagonal elements of [m] and [k] from the vector of parameters to be estimated in Phase I. In fact, this particular value was selected with the objective of achieving convergence, not necessarily with the objective of achieving a particularly

close fit of the data, or with the objective of achieving revised parameter estimates within any specified bounds. These latter two considerations may very well dictate a cutoff value for epsilon (ϵ) of somewhat less than $\epsilon = 0.2$. It is reasonable to expect that the percent error introduced in the estimates will be on the order of $100 \times \epsilon$, e.g. 20% for $\epsilon = 0.2$. As a matter of fact, in the first example of this section where an error value of .097 was calculated, response derivation of the revised analytical model from the "exact" solution was approximately 10%.

5. DEFINITION OF SUBMATRICES

Phase II of the procedure described in Section 3 uses submatrix scaling coefficients as parameters of $[M]$ and $[K]$ to be estimated. No discussion was included with regard to how the submatrices are defined. Initially, it was presumed that the analyst responsible for generating $[M]$ and $[K]$ would be able to define approximate submatrices, since the coordinate system associated with these matrices of the "dynamic model" does refer to physical displacements on the structure itself. This may still be one alternative. However, sufficient question was raised about the practical feasibility of doing this, that other alternatives must at least be considered.

5.1 Substructuring Basis

It is well known that in linear structural analysis, mass and stiffness matrices are generated by a process of superposition. When assembling these matrices from finite element contributions, the element mass and stiffness matrices are mapped into the global mass and stiffness matrices. In a similar manner, when substructuring techniques are employed, the substructure matrices are mapped into the complete structure (global) system mass and stiffness matrices.

This procedure suggests that each of these substructure matrices, after it has been mapped into global matrix elements, constitutes an appropriate submatrix for use in Phase II parameter estimation. In effect, if each substructure matrix is associated with a scaling parameter, the estimator will scale the overall stiffness and mass of the substructure up or down as required to obtain a better fit of the data. Focus, with regard to how large a portion of the structure to look

at in each case, will of course depend on the relative size of the substructure compared to the system as a whole. In theory, it should be possible to subdivide the structural system however one chooses. In a practical sense, however, this option may be precluded by having to select the substructuring for other reasons, and perhaps by having not kept the intermediate results necessary to define the required submatrices.

In conclusion, while the substructuring approach is theoretically sound and quite straightforward, it may sometimes be practically difficult to implement, particularly if one comes along after all of the modeling and condensation has been completed, and wishes to use intermediate results which are difficult to access, or not available at all. In this case, still another

5.2 Orthogonal Modes Basis

A third alternative has been postulated. Considering that all one may have to work with is $[M]$ and $[K]$ in their final form, the question is, is there any way these system matrices might be manipulated to derive appropriate submatrices for Phase II estimation.

The use of orthogonal modes provides a basis for doing so. It is well known that the natural dynamic mode shapes of a structure can be used to generate a modal expansion of both the mass and stiffness matrices. The form of such an expansion is as follows:

$$[M] = \sum_i [M] \{\phi\}_i \{\phi\}_i^T [M] \quad (5-1)$$

$$[K] = \sum_i \omega_i^2 [M] \{\phi\}_i \{\phi\}_i^T [M] \quad (5-2)$$

Assuming the orthogonality condition $\{\phi\}_i^T [M] \{\phi\}_j = [I]_{ij}$. The implication here is that each submatrix, say

$$[M]_i = [M] \{\phi\}_i \{\phi\}_i^T [M], \text{ or}$$

$$[K]_i = \omega_i^2 [M]_i$$

could be associated with a scaling parameter, β_i or α_i , such that

$$[M] = \sum_i \beta_i [M]_i$$

$$[K] = \sum_i \alpha_i [K]_i$$

In case a partial set of modes is used, we may write

$$[M] = [\bar{M}] + \sum_i \beta_i [M]_i \quad (5-3)$$

$$[K] = [\bar{K}] + \sum_i \alpha_i [K]_i \quad (5-4)$$

where $[\bar{M}]$ and $[\bar{K}]$ are defined in such a way as to complete $[M]$ and $[K]$.

This approach would conceivably work for the problem at hand, with the α 's and β 's initially set to unity. The estimator would adjust them according to the procedure described for Phase II. The only drawback to this particular approach is that it may not provide much insight into the meaning of adjustments made to elements of $[M]$ and $[K]$.

An alternative choice of modes for defining submatrices of $[K]$ is one which satisfies the orthogonality condition:

$$\{\delta\}_i^T [K] \{\delta\}_j = 0 \quad (5-5a)$$

$$\{\delta\}_i^T [K] \{\delta\}_i = K_i^* \quad (5-5b)$$

Suppose that

$$\{\delta\}_i = [K]^{-1} \{f\}_i \quad (5-6)$$

where $\{f\}_i$ is a set of forces which are in equilibrium. That is, given $\{f\}_i$ along with $[K]$, we can evaluate $\{\delta\}_i$ and K_i^* . If we generate "modal deflections"

$$\{\phi\}_i = \frac{1}{\sqrt{K_i^*}} \{\delta\}_i \quad (5-7)$$

then

$$[\phi]^T [K] [\phi] = [I] \quad (5-8)$$

and

$$[\phi]^T [K] [\phi] = [\phi]^T ([K] [\phi] [\phi]^T [K]) [\phi] \quad (5-9)$$

Thus, we may define

$$[\hat{K}] = [K] [\phi] [\phi]^T [K] = \sum_i [K]_i \quad (5-10)$$

where

$$\begin{aligned} [K]_1 &= [K] \{\phi\}_1 \{\phi\}_1^T [K] \\ &= \frac{1}{K_1^*} \{f\}_1 \{f\}_1^T \end{aligned} \quad (5-11)$$

By definition, then

$$[K] = [\bar{K}] + \sum_1 \alpha_1 [K]_1 \quad (5-12)$$

where the α 's are the parameters associated with $[K]$ to be estimated, and their initial values are

$$\alpha_1 = 1/K_1^* \quad (5-13)$$

Our objective is to define force vectors such that $\{f\}_1^T \{\delta\}_j = 0$ i.e., the virtual work done by one set of forces, on the displacements caused by another set of forces, is zero. Intuitively this will be the case, for example, when each set of forces is in equilibrium, and applied to a different part of the structure. This approach should provide a means of defining submatrices associated, therefore, with different parts of a structure.

6. CONCLUSIONS AND RECOMMENDATIONS

The following two subsections itemize specific conclusions and recommendations resulting from the study to date.

6.1 Conclusions

The formulation originally proposed for Phase I of the estimation procedure has been modified in the sense that both mass and stiffness parameters, as well as damping parameters, are now estimated. Originally, it was reasoned that by normalizing the modal vectors of the original "dynamic model" to yield an identity matrix for the mass matrix of the prior "analytical model," that only elements of the (generalized) stiffness matrix of the "analytical" model would have to be estimated. This reasoning proved to be incorrect, and the necessary revisions to the procedure were made.

Based on this modified formulation, a general computer code was developed. It is presently separated into two parts, one pertaining to the Phase I estimation procedure which refines a modal representation of the dynamic model, and the other pertaining to the Phase II estimation procedure which takes the refined modal representation and revises the mass and stiffness matrices of the original dynamic model. The two separate programs will be linked by a computer data file.

Numerous example problems have been run to test the operation of computational algorithms. While some of these examples are not realistic in the sense that they do not necessarily represent anticipated applications to real problems, they do serve to identify potential problem areas, and have been selected to test the methodology under extreme conditions. Thus, while the basic methodology has been successfully demonstrated, several related problem areas have also been illuminated.

Generally speaking, these problem areas can all be identified with one basic problem - that of determining a sufficiently good prior model from which to begin the estimation procedure. A "good" prior model must first of all embody a "proper configuration". That is, it must contain the proper elements. If the modal representation of the prior "analytical model" is deficient in the sense that critical off diagonal elements (modal coupling elements) are ignored, then computations may not converge. If the critical elements are included, but initial estimates of their values are too far off, computations may still not converge. Both of these conditions tend to be aggravated by improper estimates of the uncertainty of the various parameters. For example, if the initial estimates of these parameters are greatly in error, while corresponding uncertainties are small by comparison, the two sets of input are inconsistent with each other, relative to the target model.

As with any new tool, experience will be required to become proficient in its use. However, improvements to the methodology are suggested by the experience gained so far. In particular, the analysis presented in Section 4 has the potential of significantly improving (1) the configuration of the prior model, (2) the initial estimates of the prior model parameters, and (3) estimates of the uncertainties corresponding to these initial parameter estimates.

The following subsection itemizes specific recommendations for subsequent investigation and implementation.

6.2 Recommendations

Our first recommendation is to implement the first-order correction procedure to the prior "analytic model described in

Section 4.1. This should take advantage of test data not otherwise used explicitly in the estimation procedure, to produce a significantly improved prior model. The efficacy of this approach should be easy to demonstrate by application to some of the analytic examples presented in Section 3. This approach should provide, in addition to improved estimates of initial parameter values, a quantitative basis for evaluating parameter uncertainties based in part on measured test data.

Secondly, it is recommended that the banding criteria developed in Section 4.2 be investigated further as a means of helping to establish a proper model configuration. These criteria apply to the damping parameters, as well as the mass and stiffness parameters.

In order to maximize the utility of the procedure, it is recommended that a nested sequential estimation capability be investigated for implementation in Phase I. The intent here is to estimate parameters associated with a limited frequency band which moves incrementally across the frequency spectrum in the inner loop of the estimator, and then if necessary, sweep the frequency spectrum again with additional test data. This would in general expand the capacity for the number of parameters to be estimated.

It is recommended that a method such as that described in Section 5.2 be implemented to define appropriate submatrices for use in Phase II. This would provide a straightforward means of localizing adjustments to the dynamic model, given only the mass and stiffness matrices of that model.

Finally, it is recommended that additional analytical examples be investigated as a means of gaining experience with the modified programs prior to their application to real problems.

This experience should have a pay-off in helping to establish guidelines for all prospective users.

REFERENCES

1. Collins, J.D., Hart, G.C., Hasselman, T.K., and Kennedy, B., "Model Optimization Using Statistical Estimation," J. H. Wiggins Company Technical Report No. 73-1087-2, May 1973.
2. Collins, J.D., Hart, G.C., Hasselman, T.K., and Kennedy, B., "Statistical Identification of Structures," AIAA Journal Vol. 12, No. 2, February 1974.
3. Hasselman, T.K. and Hart, G.C., "Modal Analysis of Random Structural Systems," ASCE J. Eng. Mech. Div., EM3, June 1972, pp. 561-579.
4. Hasselman, T.K., "Modal Coupling in Lightly Damped Structures," AIAA Journal, Vol. 14, No. 11, November, 1976, pp. 1627-1628.

NOMENCLATURE

The following symbols are used in this report:

A	=	Real portion of $H(\Omega)$ - dynamic model coordinates
a	=	Real portion $\tilde{H}(\Omega)$ - analytic model coordinates
B	=	Imaginary portion $H(\Omega)$ - dynamic model coordinate
b	=	Imaginary portion $\tilde{H}(\Omega)$ - analytic model coordinate
C	=	Damping parameter - dynamic model
c	=	Generalized damping parameter - analytic model
$f(t)$	=	Forcing function in dynamic model coordinates
$g(t)$	=	Scalar function of time
$H(\Omega)$	=	Transfer function - dynamic model coordinates
$\tilde{H}(\Omega)$	=	Transfer function - analytic model coordinates
I	=	Identity
M	=	Mass parameter - dynamic model coordinates
m	=	Generalized mass parameter - analytic model
P	=	Force distribution parameter - dynamic model coordinates
$\tilde{P}$	=	Force distribution parameter - analytic model coordinates
q	=	Coordinate of modal model, time domain
R	=	Effective state variable (equation 2-1)
r	=	State variable
T	=	Sensitivity of observation quantity to change in state variable
t	=	time
u	=	Coordinate of dynamic model, frequency domain
U	=	Coordinate of dynamic model, time domain
x	=	Coordinate of analytic model, time domain
Y	=	Observation (i.e., measured) quantity
y	=	Effective observation quantity (equation 2-2)
$\tilde{Z}$	=	Variable of convenience (equations 2-10 and 2-27)
X	=	Coordinate of analytical model, frequency domain

α	=	Scaling parameters for dynamic model mass and stiffness matrices
δ	=	Kronecher delta function, also displacement vector
ϵ	=	Random error
ζ	=	Damping factor
η	=	Variable of convenience
λ	=	Eigenvalue ($\text{rad}^2/\text{sec}^2$)
ξ	=	Modal damping parameter - modal model
Σ	=	Summation
σ^2	=	Variances or covariances
ϕ	=	Modal deflection - dynamic model
ψ	=	Modal deflection - analytic model
Ω	=	Excitation frequency (rad/sec)
ω	=	Natural frequency (rad/sec)
$\hat{i}$	=	Square root of -1
[]	=	Matrix
{ }	=	Vector

Supplementary Symbols

ΔM	=	First order correction to [$^{\circ}M$]
ΔK	=	First order correction to [$^{\circ}K$]
$\Delta \lambda$	=	Difference between prior model eigenvalues and measured frequencies squared
$\Delta \phi$	=	Difference between prior model eigenvectors and measured mode shapes.
$\Delta \eta$	=	Modal contribution (prior model modes) to $\Delta \phi$
Δm	=	First order correction to [$^{\circ}m$]
Δk	=	First order correction to [$^{\circ}k$]
D	=	Denominator
$D(\hat{i}\Omega)$	=	Diagonal scaling matrix (complex)
Γ	=	Scaled coordinate of analytical model, frequency domain
α	=	Scaled coordinate of analytical model, time domain
β	=	Frequency ratio; also used to denote scaling parameter

Subscript

e	=	Desired value
g	=	Summation index
h	=	Index for state variables
i	=	Index for dynamic model coordinates - response point
j	=	Summation index, index for excitation frequencies, or mode shapes
k	=	Index for dynamic model coordinates - driven point
l	=	Index for "observation" vector
m	=	Index for analytic model coordinates, summation index
n	=	Summation index
o	=	Summation index
p	=	Prior model
q	=	Index for analytic model coordinates, summation index
r	=	Index of mass and stiffness matrices
s	=	Index of mass and stiffness matrices

Superscript (either right, left, or above)

I	=	Imaginary component of complex quantity
N	=	Total number of scaling parameters
R	=	Real component of complex quantity
T	=	Transpose of
°	=	Original dynamic model, unvarying
*	=	Complex conjugate
·	=	Differentiation
-	=	Non-varying portion of a matrix; also used to denote normalized coordinates in frequency domain
~	=	Analytic model
^	=	Approximate Value (except for $\hat{i}$)
z	=	Analytical model in scaled coordinate system

APPENDIX A

DESCRIPTION OF COMPUTER PROGRAMS

The parameter identification methodology and corresponding computer programs have been set up as a two step operation. First, a new set of generalized mass and stiffness matrices* are developed for the analytical model. Then these matrices are used as "observation" data to develop new scaling parameters for the dynamic model mass and stiffness matrices.

The two computer programs are not automatically coupled. Currently, the user must transfer the results from Phase I to cards to be used as input to Phase II. Eventually the identification programs will be configured so that disk files can be used as the transfer medium removing much of the work load from the user. Each program would still function independently, however. The advantages to keeping each phase independent are

- It may take several runs before Phase I produces a set of modes and frequencies acceptable to the engineer from his knowledge of the structure.
- The user should evaluate the Phase I results before proceeding to Phase II to insure the new model is meaningful.
- The computer programs are simpler and the core usage more efficient.

Both programs were originally dimensioned for relatively small problems to minimize computer costs during the early check-out

* The generalized mass and stiffness matrices for the dynamic model are the mass and stiffness matrices for the analytic model.

and investigation phase. They will be redimensioned in the final versions of the programs to take advantage of available computer memory core when running the practical demonstration problems.

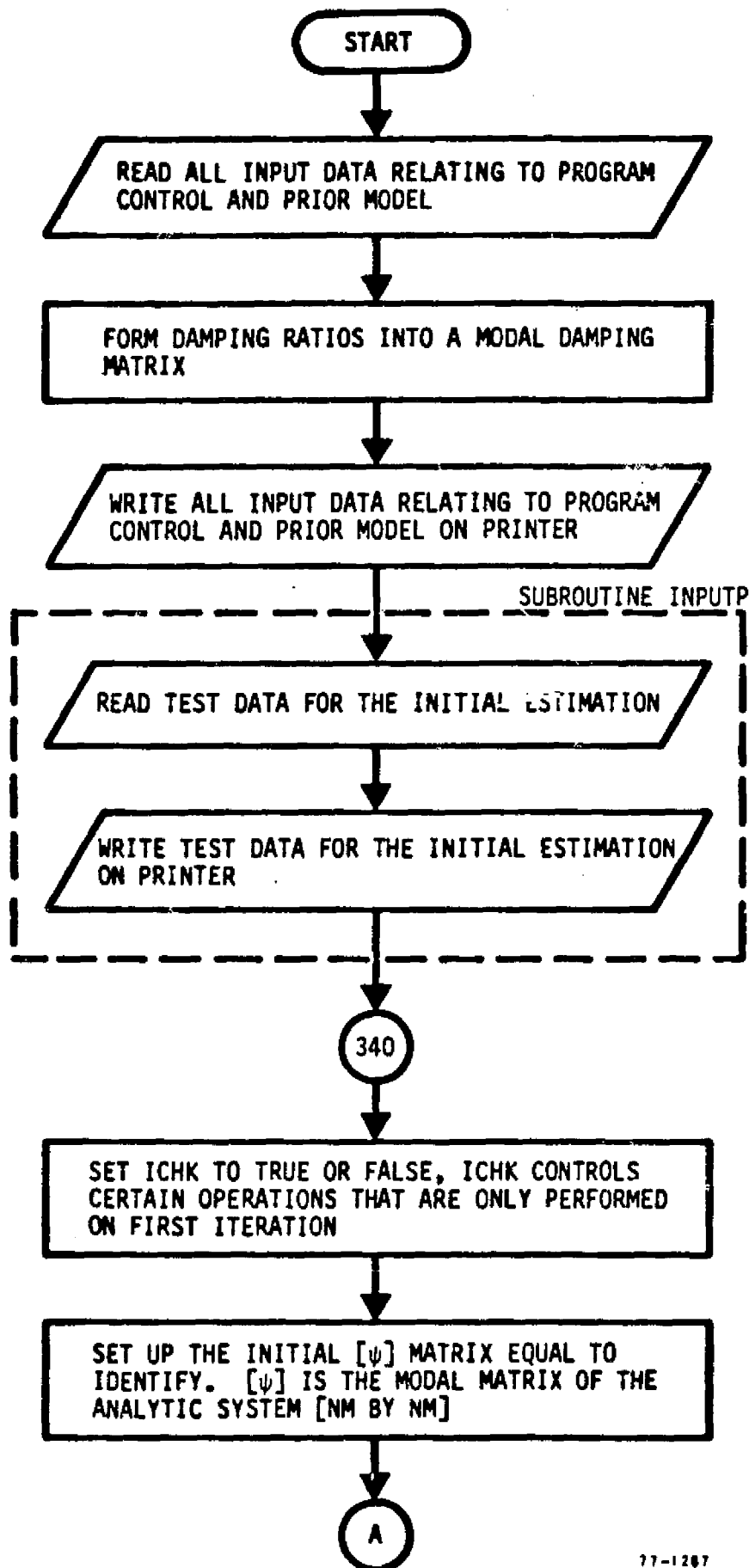
A-i Program ESTIMA (Phase I)

The Phase I procedures have been subdivided in two independent operations which are executed alternately and repeated until convergence to the test response is obtained. Phase I-A iterates on the mode shapes and natural frequencies until no further improvement can be obtained. Phase I-B then iterates on the modal damping to further improve the fit. When no further improvement is obtained here, Phase I-A is repeated. These two operations alternate until the optimum combination of modal parameters (frequencies, mode shapes, damping) is obtained.

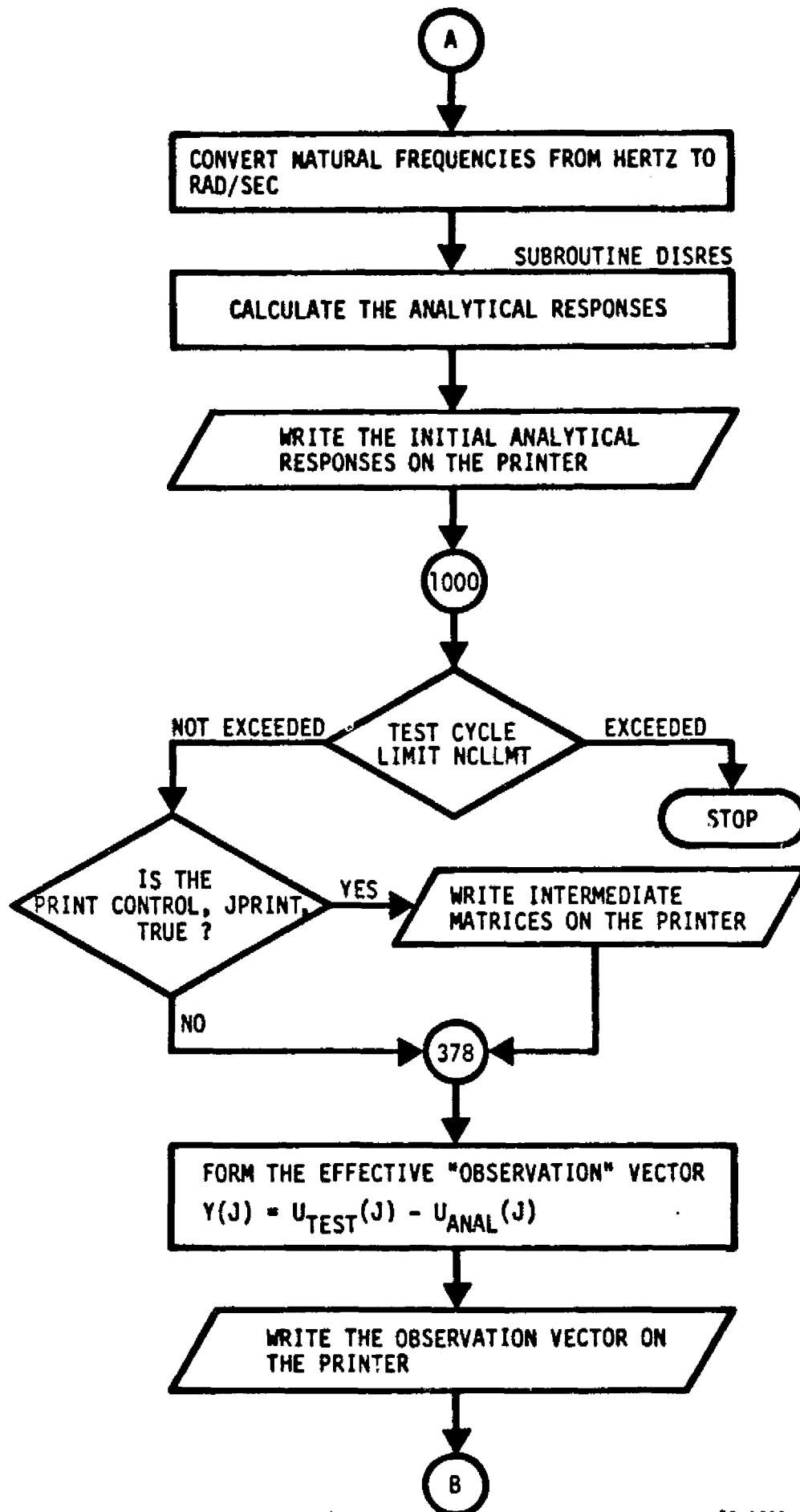
The equations needed to perform these operations have been programmed for a CDC 6600. These have been broken down into a main program plus nine subroutines. A flow chart of the main program is shown in Figure 1. As of this date the portion of the computer program specifically related to Phase I-B operations has not been entirely verified. The example problems performed to date have all been designed to develop a greater understanding of the Phase I-A methodology. They have been set up as if the damping ratios were exactly known prior to the identification process.

DISRES	Computes transfer functions for analytic model coordinates and dynamic model degrees of freedom. Computes frequency response for same two coordinate systems.
--------	---

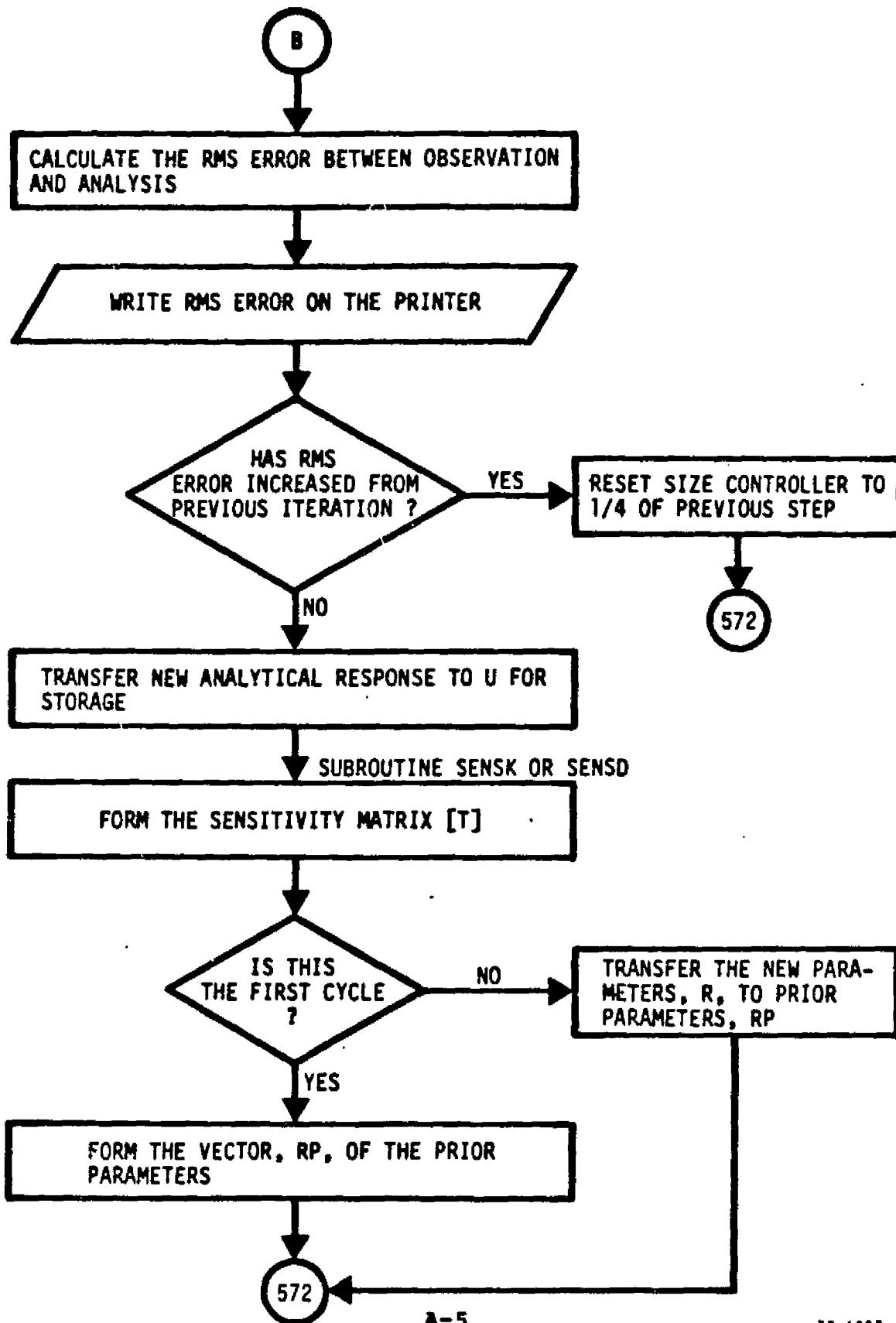
FIGURE 1. FLOW CHART FOR PROGRAM ESTIMA



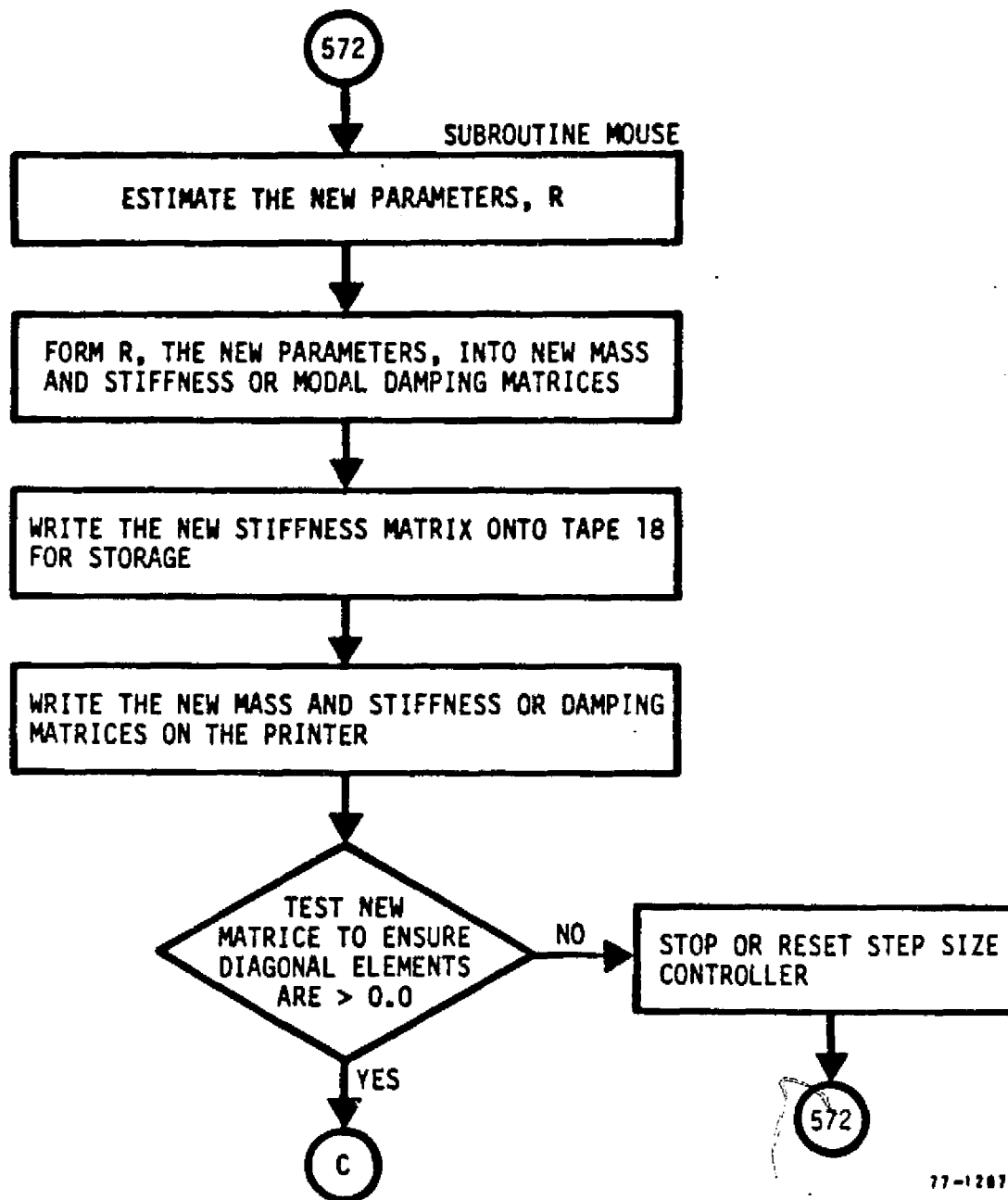
FLOW CHART FOR PROGRAM ESTIMA (Continued)



FLOW CHART FOR PROGRAM ESTIMA (Continued)

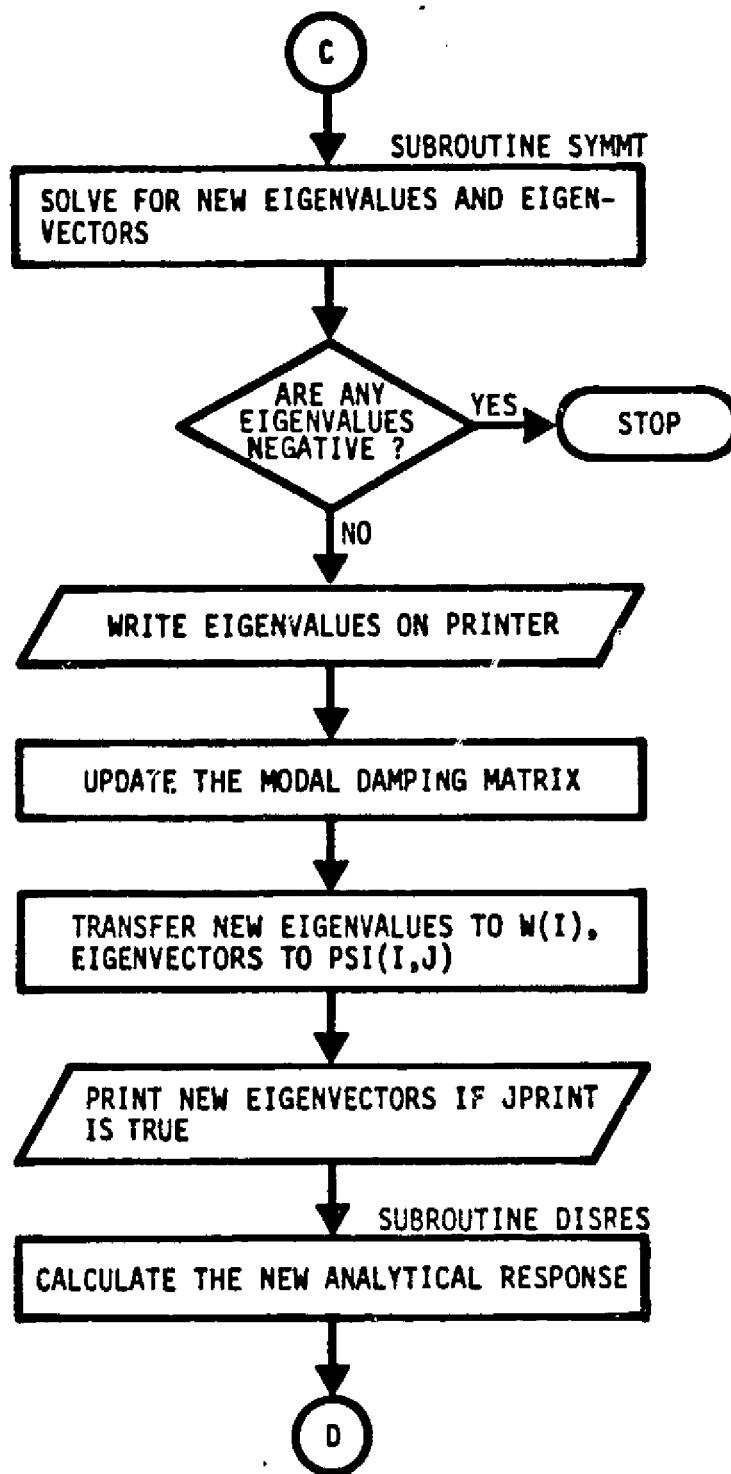


FLOW CHART FOR PROGRAM ESTIMA (Continued)



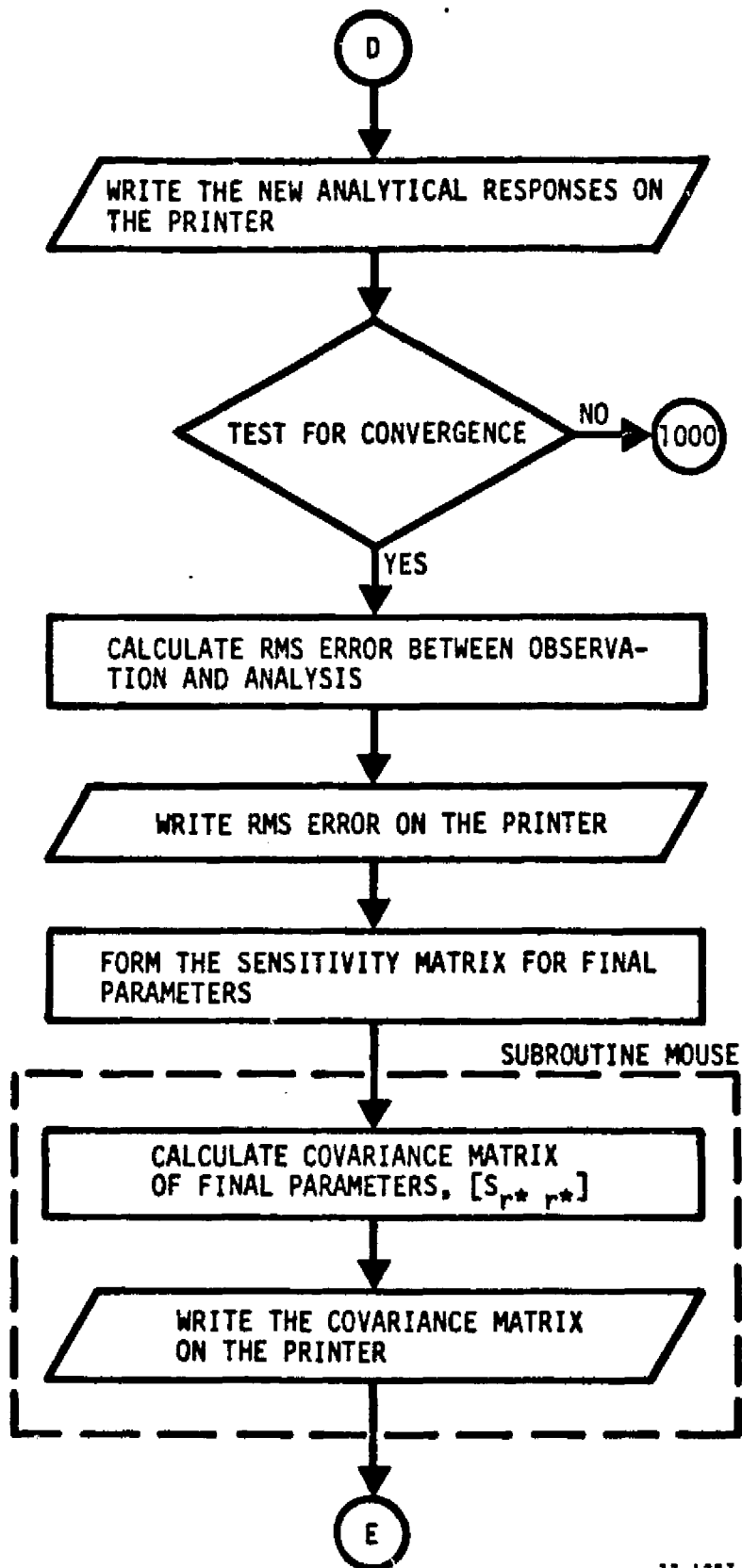
77-1287

FLOW CHART FOR PROGRAM ESTIMA (Continued)

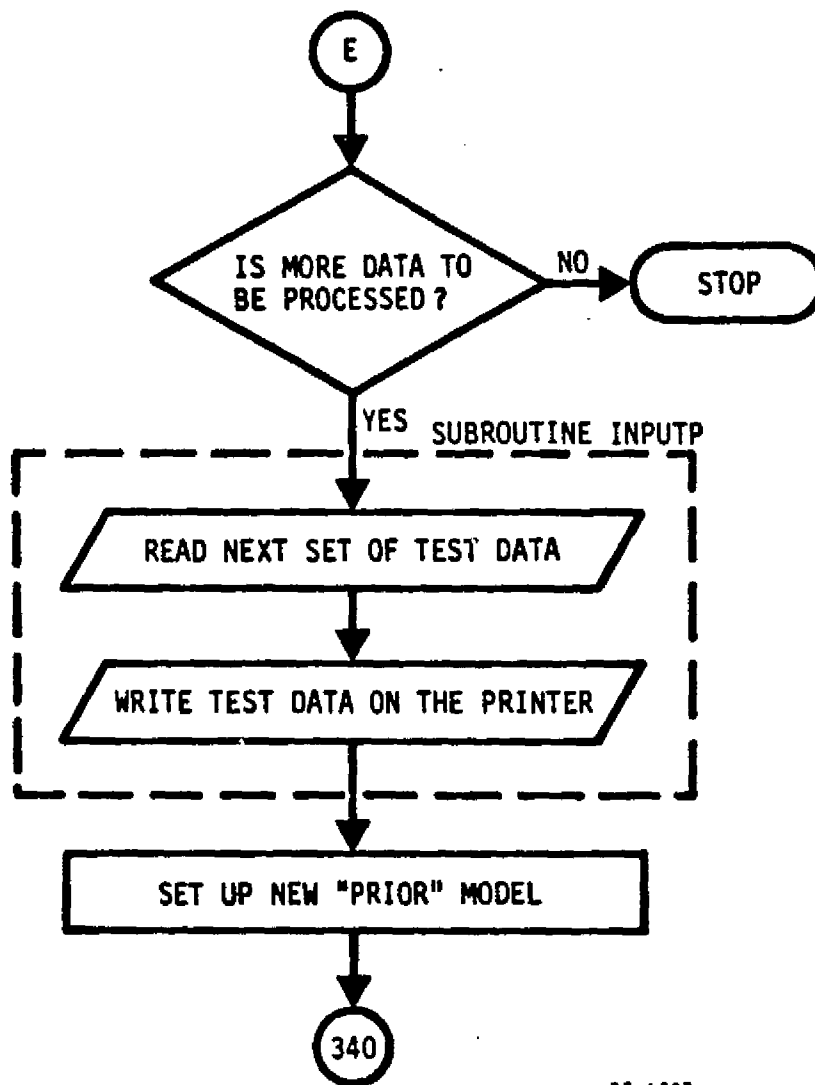


77-1267

FLOW CHART FOR PROGRAM ESTIMA (Continued)



0-2



77-1207

SENSK	Computes the sensitivity matrix for analytic model mass and stiffness matrices (i.e., the sensitivity of the frequency response with respect to variations in the analytical model mass and stiffness matrix).
SENSD	Computes the sensitivity matrix for modal damping matrix.
MOUSE	Estimates a new analytic model stiffness matrix. Also develops a new covariance matrix when each set of iteration cycles is complete. Based on same equations as computer program MOUSE [2].
GIVHO	Uses Householders method to reduce a real symmetric matrix to tridiagonal form. Isolates eigenvalues and eigenvectors. Adapted subroutine from program MOUSE [2].
INVERT	Inverts a real symmetric matrix. Adapted subroutine from program MOUSE [2].
INVECC	Inverts a complex symmetric matrix.
MATMUL	Multiplies tow conformable matrices. Adapted subroutine from program MOUSE [2].
MATOUT	Prints all non-zero elements of a matrix. Adapted subroutine from program MOUSE [2].

INPUT DATA FOR PROGRAM ESTIMA

Card One

Cards 1 to 3 provide information on input-output requirements, on the convergence criteria, and the elements to be estimated.

IPRINT, JPRINT		

L10	, L10	FORMAT

where

IPRINT	is a flag which controls printing of the modal orthogonality check.
	=T print the entire orthogonality check
	=F print only bad elements of orthogonality check
JPRINT	is a flag which controls printing of the various intermediate matrices
	=T print intermediate matrices
	=F do not print intermediate matrices

Card Two

NCLLMT, NB		

I5	, I5	FORMAT

where

NCLLMT	is the maximum number of iterations allowed
NB	is the bandwidth within which new elements are to be estimated.

Card Three

CONLMT, CONLM2

F10.0 , F10.0

FORMAT

where

CONLMT is a convergence criterion (ratio between calculated and observed response that will constitute success).

CONLMT is a second convergence criterion (ratio between two successive calculations that will constitute success).

Card Four

Cards 4 to 11 provide data on the prior model.

NM , NC

I5 , I5

FORMAT

where

NM is the number of modes being used.

NC is the number of degrees of freedom in the u coordinate system.

Card Five

W(I), I = 1 NM

8F10.0

FORMAT

where

NDMPFL is the control flag which specifies the type of damping information to be read.
=0 read the critical damping ratios for the NM modes of the prior model
=1 read the full NM by NM damping matrix for the prior model

Card Seven (if NDMPFL = 0)

ZET(I), I = 1, NM

8F10.0

FORMAT

where

ZET(I) are the critical damping ratios for the NM modes of the prior model being used for this estimation.

Card Seven (if NDMPFL = 1)

(ETA(I,J), I = 1, NM), J = 1, NM

8F10.0

FORMAT

where

ETA(I,J) are the NM times NM elements of the modal damping matrix for the prior model being used for this estimation.

Card Eight

(PHI(I,J), I = 1, NC), J = 1, NM

8F10.0

FORMAT

where

PHI(I,J) are the elements of the modal matrix
for the NM modes being used for this
estimation.

Card Nine

SRPRP(I), I = 1, NP

8F10.0

FORMAT

where

SRPRP(I) are the initial variances of the elements
of the generalized mass matrix that are
to be estimated. The program determines
NP based on the band width specified
and the number of modes. Only the diagonal
and upper right elements are estimated.

Card Ten

SRPRP(I), I = NP+1, 2*NP

8F10.0

FORMAT

where

SRPRP(I) are the initial variances of the elements
of the generalized stiffness matrix
that are to be estimated. These elements
correspond to the elements of the mass
matrix being estimated.

Card Eleven

SRPRPD(I), I = 1, NP

8F10.0

FORMAT

where

SRPRPD(I) are the initial variances of the generalized damping matrix. These elements correspond to the elements of the mass matrix being estimated.

Card Twelve

Cards 12 to 14 provide the observation data (i.e., test data). Successive sets of observation data may be processed.

NF		

I5		FORMAT

where

NF is the number of excitation frequencies for which observation data is being read.

Card Thirteen

FQ(I), I = 1, NF		

8F10.0		FORMAT

where

FQ(I) are the observation frequencies (Hz).

Card Fourteen (a block of cards)

This block of cards must be provided for each of the NF observation frequencies. L is the index of the observation frequencies.

Card 14-A

NCT(L) , NCF(L)		

15	, 15	FORMAT

where

NCT is the number of coordinates with observation data.
NCF is the number of coordinates being forced (i.e., with shakers).

Card 14-B

ICT(I,L) , I = 1, NCT(L)		

16I5		FORMAT

where

ICT(I,L) are the locations (i.e., the degrees of freedom) of the observation data (i.e., the measured response).

Card 14-C

ICF(I,L) , I = 1, NCF(L)		

16I5		FORMAT

where

ICT(I,L) are the locations (i.e., the degrees of freedom) of the coordinates being forced.

Card 14-D

PR(I,L) I = 1, NCF(L)

8F10.0

FORMAT

where

PR(I,L) are the real components of the excitation forces.

Card 14-E

PI(I,L), I = 1, NCF(L)

8F10.0

FORMAT

where

PI(I,L) are the imaginary components of the excitation forces.

Card 14-F

UTEST(I), I = 1, NO

8F10.0

FORMAT

where

UTEST(I) are the observed responses arranged as followed:

UTEST(1) = location 1, freq. 1

UTEST(2) = location 1, freq. 2

through all observation frequencies.

Repeat for all locations.

Card 14-G

SEE(I), I = 1, NO

8F10.0

FORMAT

where

SEE(I) are the variances of the observed
response arranged as described above.

A-2 Program ESTIMB (Phase II)

ESTIMB, the Phase II, minimum variance estimator consists of a main program plus five subroutines. A flow chart of the main program is shown in Figure 2. The five subroutines are:

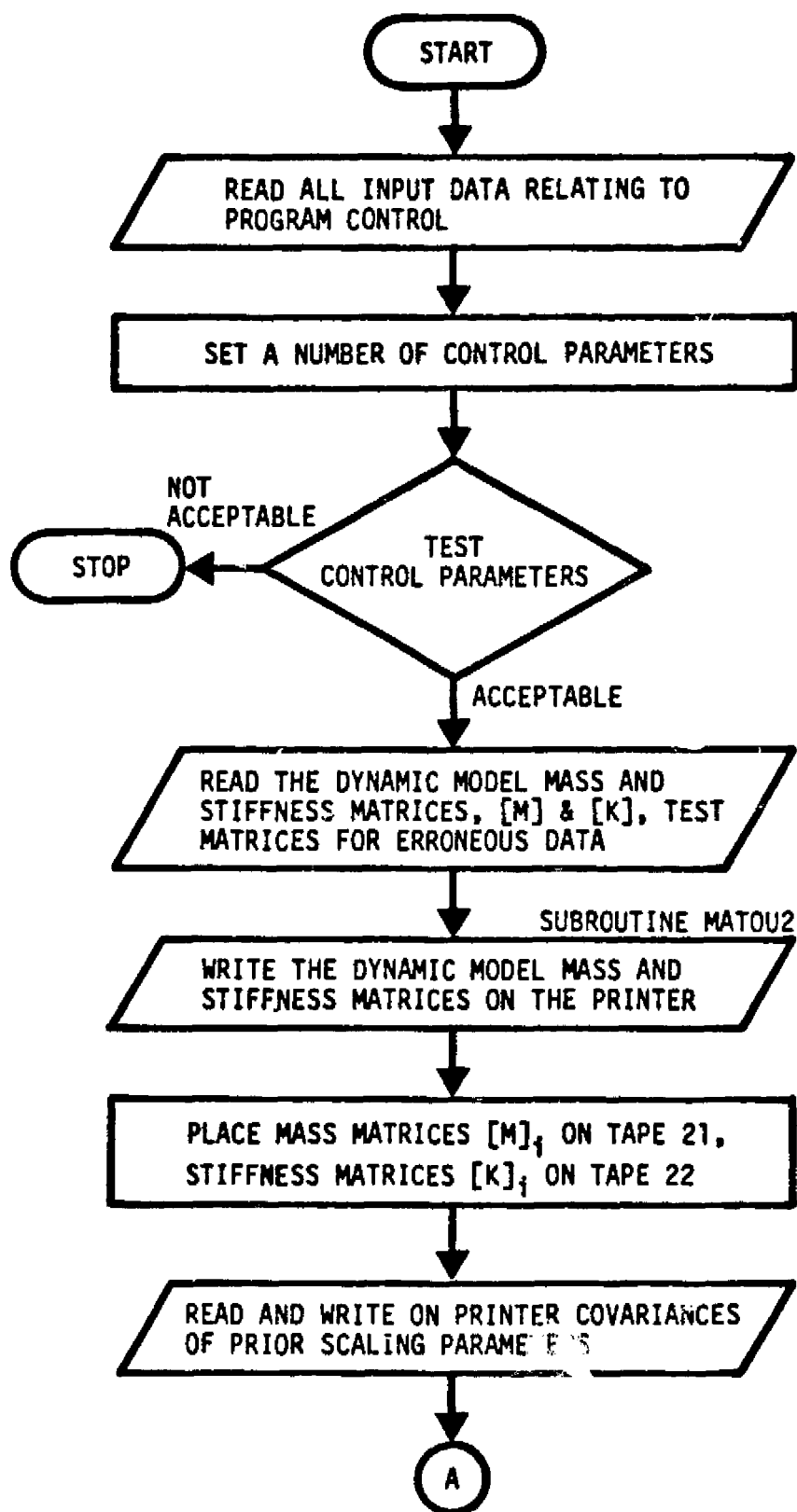
ESTIMB	Main program. Estimates a new dynamic model (mass and stiffness matrices) from the refined parameters of the analytical model (from Phase I).
MOUSE	Estimates new parameters (in this case the α_j). Provides the covariance matrix of the new parameters. Identical to the MOUSE subroutine used for Phase I.
MMULRR	Performs matrix multiplication of two real matrices without destroying either one.
MATOUT	Prints all non-zero terms of a matrix with pages, page headings, and matrix identification. Identical to the MATOUT subroutine used for Phase I.
MATOU2	Prints all non-zero terms of a matrix without pages, page headings, or a matrix identification.
INVERT	Inverts a real symmetric matrix. Identical to the INVERT subroutine used for Phase I. Uses the Choleski SDS decomposition method.

Input Data for Program ESTIMB

Card One

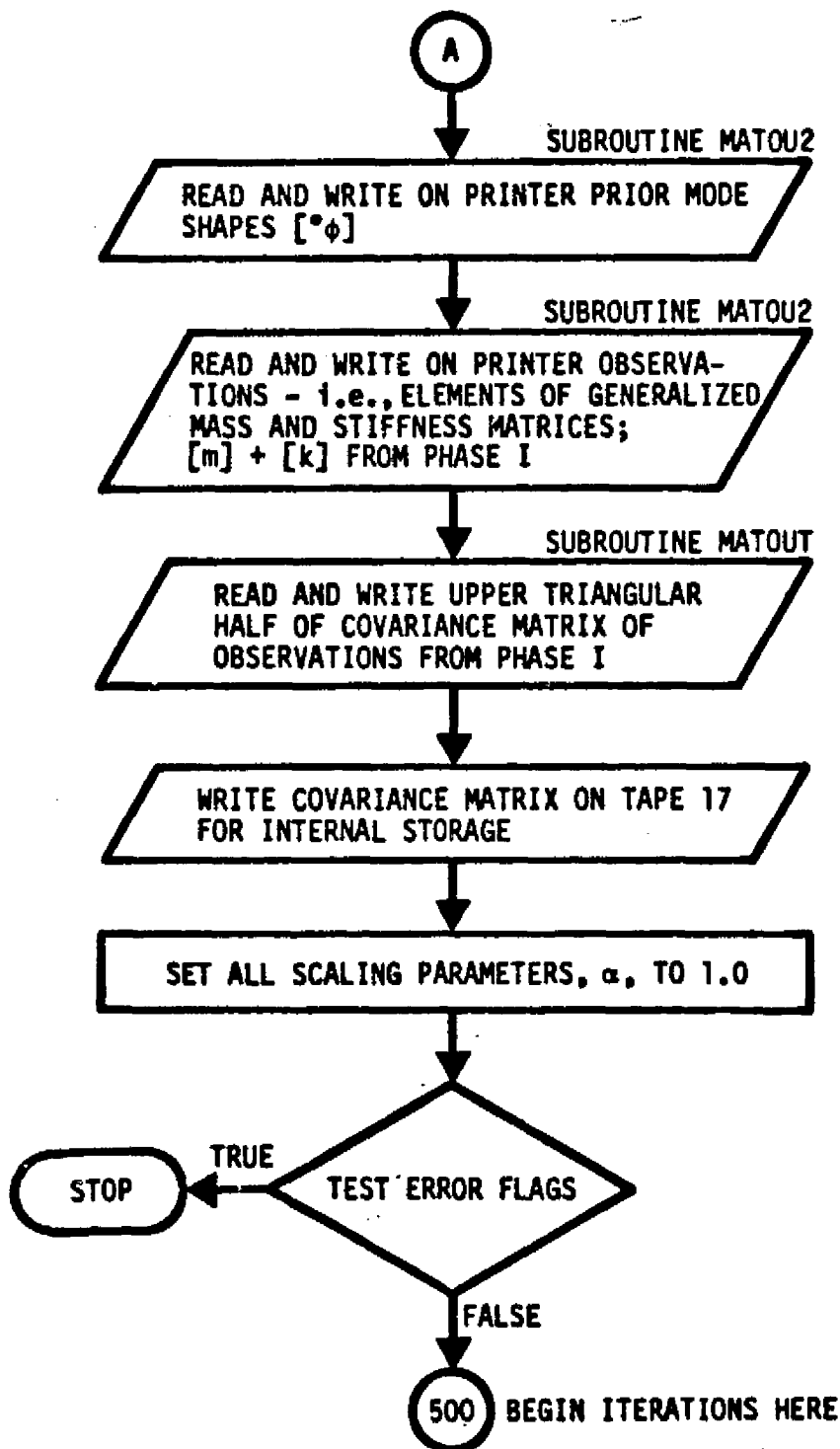
Cards 1 to 5 provide information on input-output requirements, on the convergency criteria, and on the number of elements to be estimated.

FIGURE 2. FLOW CHART FOR PROGRAM ESTIMB



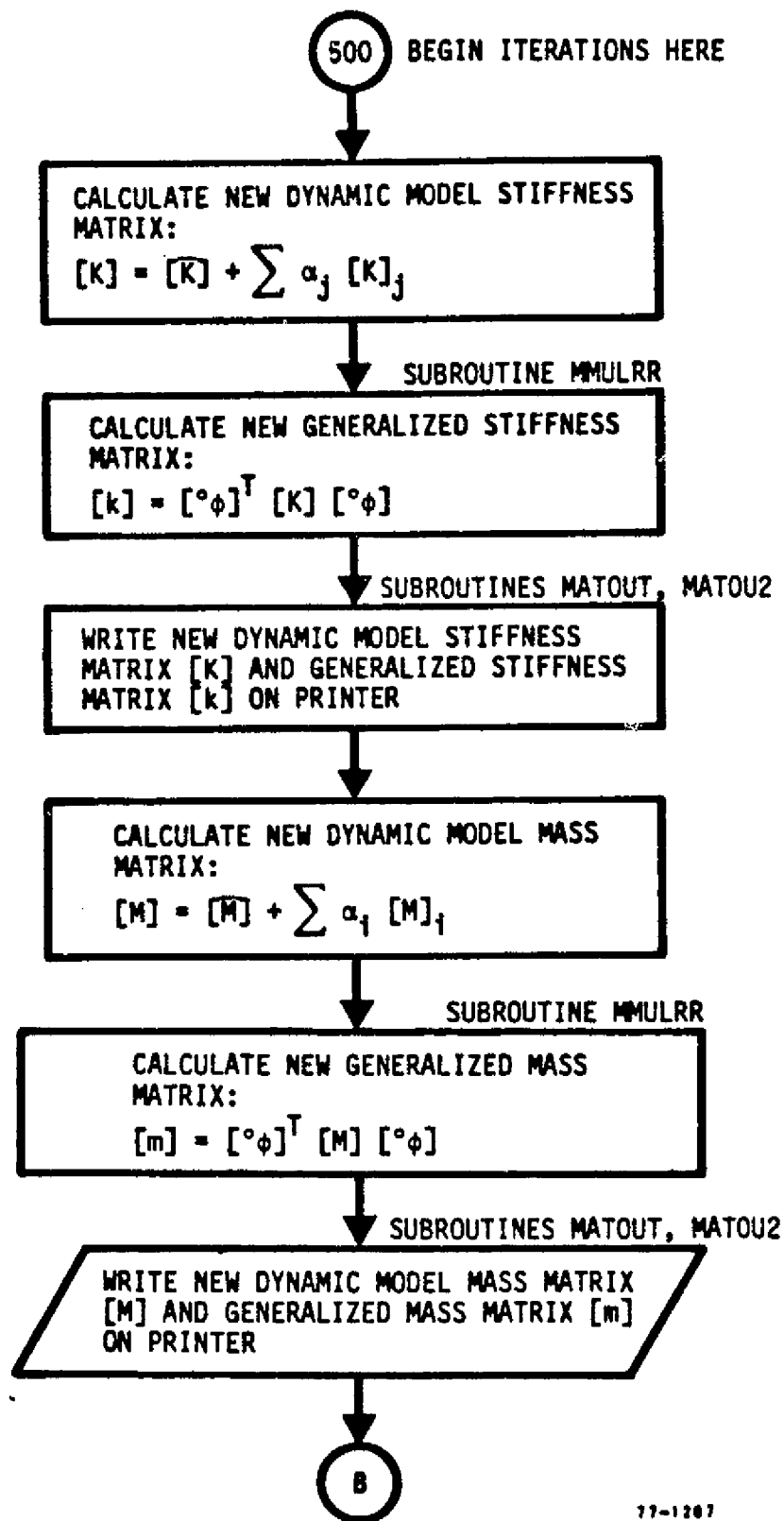
77-1287

FLOW CHART FOR PROGRAM ESTIMB (Continued)



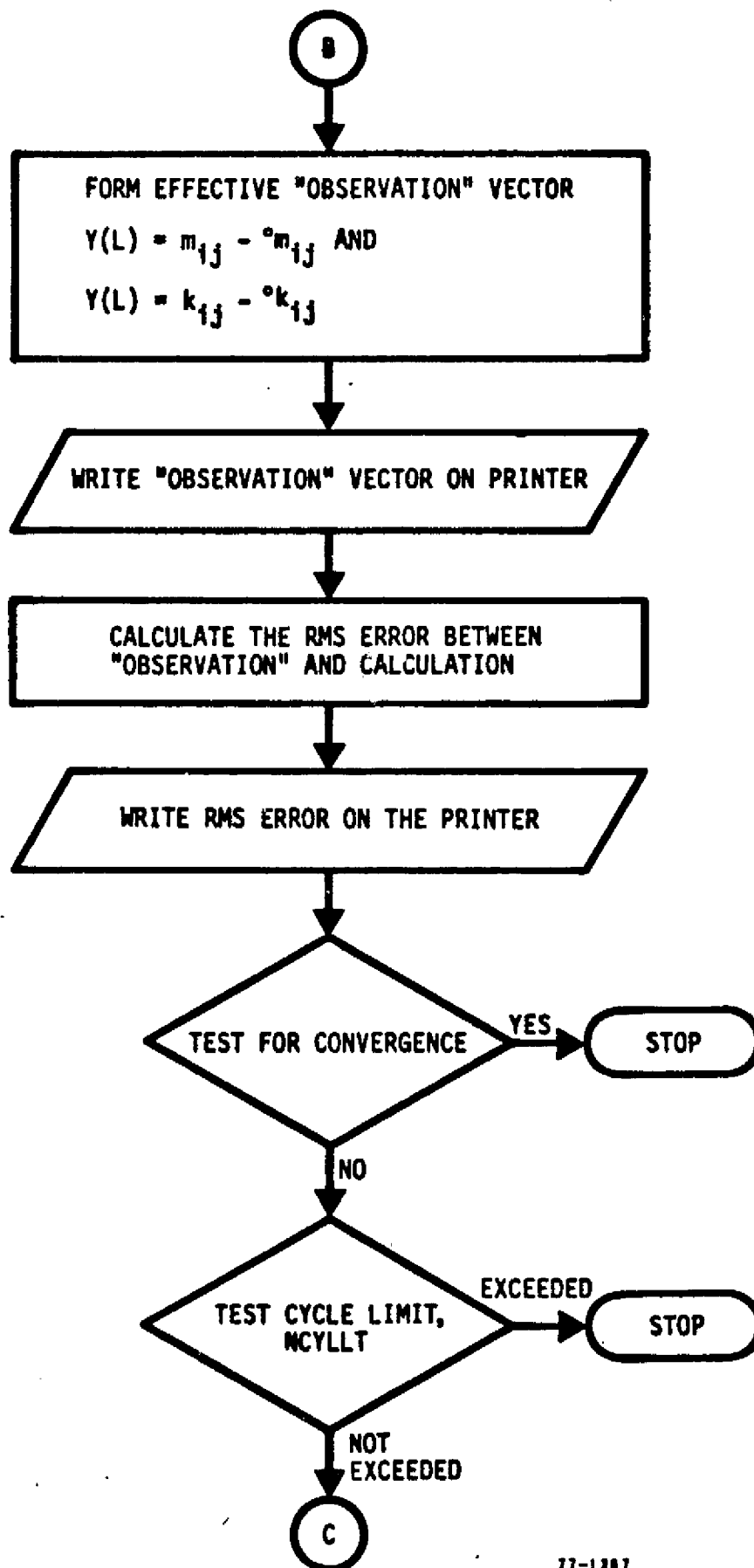
77-1287

FLOW CHART FOR PROGRAM ESTIMB (Continued)

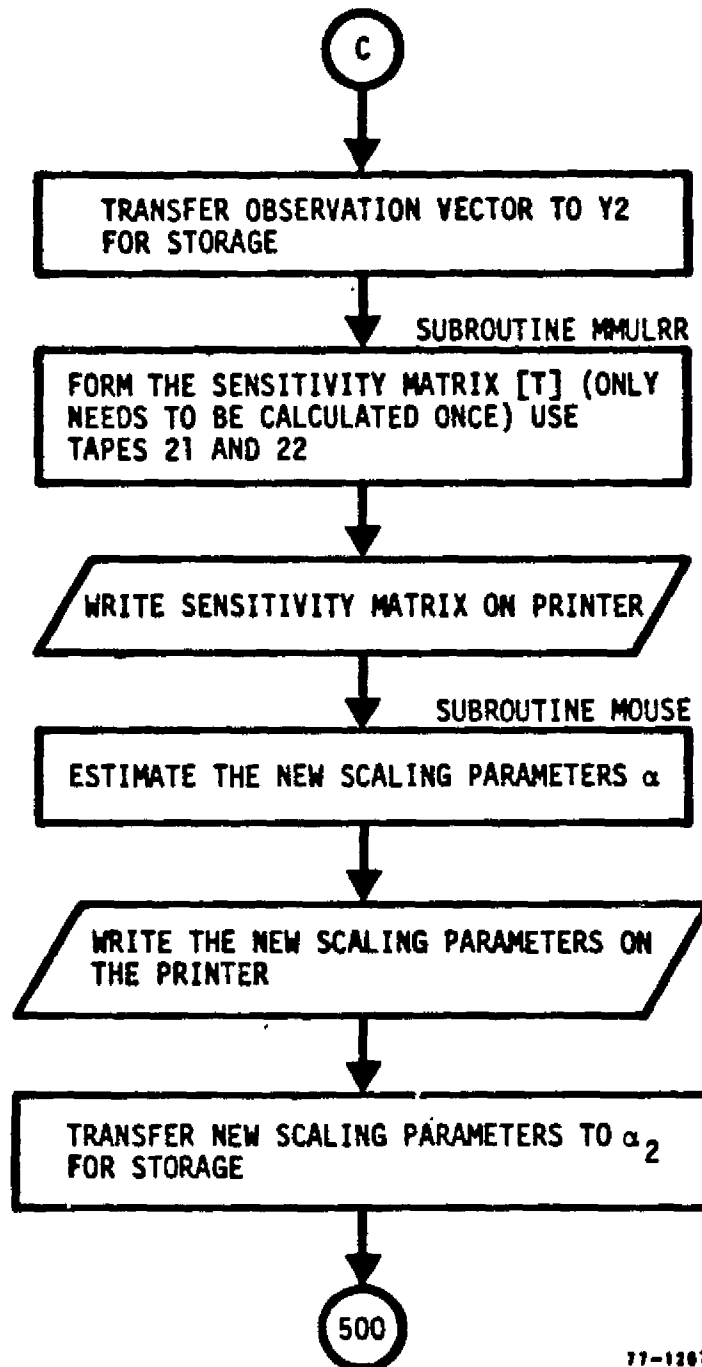


77-1207

FLOW CHART FOR PROGRAM ESTIMB (Continued)



FLOW CHART FOR PROGRAM ESTIMB (Continued)



77-1287

JPRINT

L10

FORMAT

where

JPRINT is a flag which controls printing of intermediate operations.
=T print intermediate matrices.
=F do not print intermediate matrices.

Card Two

NCYLLT

I5

FORMAT

where

NCYLLT is the maximum number of iterations allowed.

Card Three

CONLMT, CONLM2

F10.0 , F10.0

FORMAT

where

CONLMT is a convergence criterion (ratio between calculated and observed values that will constitute success) for stiffness matrix.

CONLM2 is a second convergence criterion (ratio between two successive calculated values that will constitute success) for stiffness matrices.

Card Four

CONLM3, CONLM4

F10.0 , F10.0

FORMAT

where

CONLM3 is a convergence criterion (ratio between
calculated and observed values that will
constitute success) for mass matrix.

CONLM4 is a second convergence criterion (ratio
between two successive calculated values
that will constitute success) for mass
matrix.

Card Five

NK, NM, NS

I5, I5, I5

FORMAT

where

NK is the number of K-matrix scaling para-
meters to be estimated.

NM is the number of M-matrix scaling para-
meters to be estimated.

NS is the size (degrees-of-freedom) of the
dynamic model.

Card Six

Cards 6 and 7 are repeated NK times. Cards 6 to 12 provide
information on the prior model.

N	
I5	FORMAT

where

N is the index number of the K-matrix portion to be read next.

Card Seven

JJ, KK, K(JJ, KK)	
4(I5, I5, F10.0)	FORMAT

where

JJ is the first index of the element.
 KK is the second index of the element.
 KK must $\geq$ JJ.
 K(JJ, KK) is the JJ, KK element of the portion of K-matrix being read.

Values of JJ, KK, and K(JJ, KK) are read until a value of 0 is read for JJ. Use as many cards as necessary with 4 elements per card.

Repeat cards 6 and 7 until all portions of the prior K-matrix are read in. The program will continue to read K-matrix blocks until a value of 0 is read for N. Only the upper right-hand elements of the K-matrices must be read.

Card Eight

Cards 8 and 9 are repeated NM times.

N	

I5	FORMAT

where

N is the index number of the M-matrix portion to be read next.

Card Nine

JJ, KK, M(JJ, KK)	

4(I5, I5, F10.0)	FORMAT

where

JJ is the first index of the element.
 KK is the second index of the element.
 KK must $\geq$ JJ.
 M(JJ, KK) is the JJ, KK element of the portion of the mass matrix being read.

Values of JJ, KK, and M(JJ, KK) are read until a value of 0 is read for JJ. Use as many cards as necessary with 4 elements per card.

Repeat cards 8 and 9 until all portions of the prior M-matrix are read in. The program will continue to read M-matrix blocks until a value of 0 is read for N. Only the upper right-hand elements of the M-matrices must be read.

Card Ten

SRPRP(I), I = 1, NP

8F10.0

FORMAT

where

SRPRP are the variances of the original scaling parameters. $NP = NM + NK - 2$.

Use as many cards as necessary to read all of the variances. A maximum of 20 values are presently allowed: 10 for K-matrix parameters, 10 for M-matrix parameters.

Card Eleven

ND

I5

FORMAT

where

ND is the number of modes of the dynamic model that are being used.

Card Twelve

PHIP(J,I), I = 1, NS

8F10.0

FORMAT

where

PHIP(J,I) is the i^{th} element of the j^{th} mode shape.

Use as many cards as necessary to complete each mode shape.
Repeat Card 12 for each mode shape. A maximum of 30 modes with
30 elements per mode may be read.

Card Thirteen

Cards 13 to 15 provide the "test" data.

NO	-----
I5	FORMAT

where

NO is the nubmer of "observations"

Card Fourteen

IO, JO, KO, MO	-----
2(I5, 15, E15.9, E15.9)	FORMAT

where

IO is the i subscript.

JO is the j subscript.

KO is the i, j element observation general-
ized stiffness matrix [k].

MO is the i, j element of the observation
generalized mass matrix, [m].

Repeat Card 14 as often as necessary to read all of the "test"
data. A maximum of 30 elements may be used.

Card Fifteen

JJ, KK, SEE(JJ, KK)

4(I5,I5, E10.4)

FORMAT

where

JJ is the first index of the element.
KK is the second index of the element.
 KK must $\geq$ JJ.
SEE(JJ, KK) is the JJ, KK element of the covariance
 matrix of the "observation" data.

Only read the upper right-hand elements of the covariance
matrix.